Summary of Vortex Methods Literature
(A living document rife with opinion)

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Contents

Chapter 1

Introduction

The goal of much of computational fluid dynamics is the numerical approximation of fluid flow governed by the Navier-Stokes equations. For incompressible flow, the five equations are

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1.1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \boldsymbol{g} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} \tag{1.2}$$

$$\rho \frac{\partial e}{\partial t} = \varepsilon_v + k\nabla^2 T + q_H. \tag{1.3}$$

Obviously, solving such equations requires one to calculate and track the three components of velocity $\boldsymbol{u}$, and the scalar values for pressure p, density ρ, and sometimes temperature T of a fluid over the entirety of the field for the duration of the simulation. The kinematic viscosity ν and coefficient of thermal conductivity k are assumed to be constant. Lastly, q_H represents heat sources other than conduction, and ε_v is the heat created by viscous diffusion.

1.1 Advantage of vorticity variables

In the above formulation, the pressure term must be solved by a separate elliptic equation, and the convection term often imposes severe time step limits in order to maintain stability. An alternative approach is to consider the problem in vorticity variables. The vorticity $\boldsymbol{\omega}$ is defined as

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}. \tag{1.4}$$

Because rewriting equation (1.2) to use vorticity variables does not remove its dependence on velocity, we will need to discuss the matter of recovering velocity information from the vorticity field. There are a number of conditions which are sufficient to allow the inversion to uniquely determine the velocity. If these are satisfied, the velocity field can be reconstructed uniquely by adding an integration over the vorticity support to an irrotational vector field.

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{4\pi}\int \frac{\boldsymbol{\omega}(\boldsymbol{x}',t)\times(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,d\boldsymbol{x}' + \nabla\Phi \tag{1.5}$$

The first term in the above equation is the Biot-Savart law, which describes the velocity field induced by the vorticity. The second component is a correction to the velocity caused by boundary surfaces in the flow. Section 2 will cover, in detail, the various methods used to recover the velocity field. Details of the Lagrangian discretization technique appear in section 2.1. Methods for fast evaluation of the Biot-Savart component shall be discussed in section 2.2. Computation of the $\nabla\Phi$ term typically requires the use of boundary integral methods, which are covered in section 2.3.2.

Taking the curl of the momentum equation (1.2) produces Helmholtz's vorticity equation

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} + \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nabla\times\boldsymbol{F} + \nu\nabla^2\boldsymbol{\omega}, \tag{1.6}$$

where $\boldsymbol{F}$ represents a conservative force. This formula governs the evolution of vorticity in an incompressible fluid flow, and will be one of the key equations in Lagrangian vortex methods. Section 3 of this paper will cover the vorticity equation in detail.

The first term on the right-hand side represents the change in vorticity due to vortex stretching. This term is discussed in section 3.1. The term containing ρ and p is zero in any barotropic fluid, but is important in multi-fluid simulations, and will be more thoroughly examined in section 3.5. The conservative force term $\nabla\times\boldsymbol{F}$ can be used to account for gravity and for two-way coupling in particle-laden flows. Its use will be briefly examined in section 3.7. Lastly, the final term represents viscous diffusion of vorticity, which will be discussed in section 3.2. Other terms that can appear in this equation represent the effects of surface tension (section 3.6), frame rotation (section 3.8), combustion (section 3.9.6), and compressibility (section 3.10).

A summary of these, and other, advantages of the vorticity-velocity formulation is due to Speziale [1]. Other monographs on vorticity range from

Poincaré's aging [2] to Saffman's technical [3] to Lugt's informational [4] to Schwenk's philosophical [5]. Applications of vortex theory occur throughout fluid dynamics, and the theories have even extended to a theory of matter [6].

1.2 Advantage of Lagrangian methods

Two major forms of computational methods for fluid dynamics exist, each named after the form of the convection equations that they use: Eulerian and Lagrangian. (mathematical description)

Early work in incompressible Eulerian methods emphasized the velocity-streamfunction or vorticity-streamfunction equations, and there remain several advantages of that approach. Harlow and Welch [7] find that free-surface flows can be more easily described by velocity-pressure variables.

Low-resolution Eulerian methods, despite their inherent numerical diffusion in the convection operator, can still maintain tightly- wound vortexes with the use of "vorticity confinement" [8], an extra term in the Euler equations.

1.2.1 Front-tracking vs. front-capturing

In addition to describing the convective formulation, "Eulerian" and "Lagrangian" can describe the method by which a fluid interface is described. "Front-capturing" methods use only grid information to determine the configuration of a fluid or fluid interface. In this camp are methods such as Level Sets, (what else?)

"Front-tracking" methods (see section 3.9.1) use connected Lagrangian markers to explicitly track a fluid interface throughout its motion.

Harlow and Welch's *marker-and-cell* [7] is probably the first use of a combination of the two methods. In their method, Lagrangian particles determine which cells in a free-surface calculation contain fluid and which do not.

1.3 Advantages of vortex methods

Because many fluid dynamic phenomena of practical interest are essentially incompressible, constant-temperature, single-phase turbulent flows, their gov-

erning equations of motion are subject several simplifications, not the least of which is that the Navier-Stokes equations can be rewritten in terms of the vorticity. As mentioned above, taking the curl of the Navier-Stokes equation (1.2) gives the vorticity transport equation, simplified here as

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega} \tag{1.7}$$

Two of the benefits of this formulation are the absence of the pressure term and the automatic satisfaction of the continuity equation. The equations are now only dependent on vorticity and velocity. This greatly simplifies some numerical methods designed to solve the Navier-Stokes equations.

A vortex method, then, is characterized by both the use of the Navier-Stokes (or Euler) equations in vorticity-velocity form, and by a Lagrangian discretization of the vorticity.

In flows with minimal viscous diffusion, another advantage of the vorticity formulations manifests. In these flows, the volume of fluid with significant vorticity magnitude is typically a small fraction of the total flow volume. This means that the flow can be represented in a more compact form by vorticity than is possible with velocity. This fact lends support to computational methods in vorticity variables.

Cottet [9] states that the Lagrangian form avoids the explicit discretization of the convective term in the Navier-Stokes equations, and its associated stability constraints.

Cottet [10] uses a stability criterion for his simulations of homogeneous turbulence that is equal to $\Delta t = \|\omega\|^{-1}$. Normal finite-difference methods are usually limited to a advective CFL number of 1.0. Look in that, and other summary references for good arguments for choosing a Lagrangian approach instead of finite-difference or spectral methods for solving unsteady convection-diffusion problems in 3-space. Explicit treatment of diffusion also requires enforcement of a diffusion CFL condition, but at high Reynolds numbers, this is less restrictive than the advection CFL [11].

A problem associated with vortex methods is the difficulty in which physical degrees of freedom in the interior of fluids are dealt with. One can argue, though, that in many flows of engineering interest, the properties of the flow are constant in large regions and only rapidly changing in small, compact regions. These large regions, thus, do not have the degrees of freedom brought on by local changes in fluid properties.

Comparisons with Eulerian finite-difference schemes has shown that vortex methods can be faster by up to an order of magnitude, even when the volume is completely filled with vorticity [11]; most of the benefit being due to longer time steps allowed by the increased stability of the Lagrangian vortex method.

MOVE THIS A method for particle-grid decomposition is presented in [11]. This merges finite-difference Eulerian and standard VIC vortex methods into the same solution scheme.

An good introduction to vortex methods is given by Leonard [12]. Chen [13] presents a summary of the benefits of vortex methods.

1.4 Other Lagrangian methods

Many Eulerian (grid-based) calculation methods take advantage of the beneficial properties of Lagrangian discretizations for portions of their work. The Front-Tracking Method uses a Lagrangian mesh to track a surface of interest within the context of an Eulerian solution. Methods called "smoothed particle hydrodynamics" (SPH) can be used to study a variety of flow and flow/solid systems. They have been adapted for incompressible fluid dynamics and free surfaces [14]. Moving particle semi-implicit (MPS) methods use locally-interacting particles to compute all manners of flows: multiphase, solid-liquid interaction, free surface, etc.

Chapter 2

Velocity Field Calculation

Mention importance of velocity to advect marker points and time discretization.

2.1 Discretization of vorticity

Vorticity can be discretized in a number of ways. The first four sections describe the most frequently-used discretizations. Keep in mind that particles and segments in 2D and axisymmetric simulations correspond, respectively, to filaments and sheets in fully 3D models.

Also keep in mind (maybe create a section for this?) is that remeshing of all discretizations can be done in a local sense (point insertion, others) or a global sense (radial basis functions, global cubic splines, Cottet's particle weight scheme, even Grant's Delaunay triangulations!).

2.1.1 Vortex particle methods

Rosenhead [15] wrote an expression for a desingularized vortex particle in 1930 in his study of the stability of a double row of vorticies. This was inspired by Kelvin's papers. The first true dynamical vortex simulation was done by Rosenhead [16] in two dimensions with singular point vorticies, later repeated by several authors [17, 18], and repeated with regularized vortex cores by Chorin and Bernard [19] and Kuwahara and Takami [20]. A summary of the initial study of the vortex sheet is contained in Zalosh [21].

Note that desingularizing the Biot-Savart equation is not the same as

using finite-cored vortex blobs, though both achieve the same effect—making the vortex sheet well-posed. Leonard [12] discusses vortex blob core functions for 2D particle methods.

As listed in [22], Tung and Ting [23] and Saffman [24] found that the distribution of vorticity across the core of a viscous vortex ring with small cross-section is Gaussian.

A numerical study of the 2D Kelvin-Helmholtz instability [25] shows that core overlap is desirable. An in-depth study of singular and desingularized vortex particles, and an introduction to a new regularization for vortex blobs—the "δ-equations"—appears in Krasny [26] and Rottman and Stansby[27].

Chorin's [28] presents 3D vortex blob calculations, though stretch was computed using a local segment approximation, making it essentially a vortex filament method. Leonard [12] concurrently proposed a 3D filament method. Beale and Majda [29] first proposed using spherical particles for 3D computations, using Lagrangian update to compute stretch, and forgoing all element connectivity information. Until then, all 3D vortex methods used filaments. The first 2D proofs were by Hald and Del Prete [30] and Hald [31]. Proof was provided for the existence of a solution for short times, as long as the overlap between the vortex blobs remained [29, 32]. Beale [33] gives a convergence proof of the disconnected, discrete vortex method by requiring that the vortex core radius be larger than the interparticle spacing. It took several more years for the first implementations of the disconnected particle vortex method [34, 35] in three dimensions to appear. Since then, it has become the predominant form of modern computational vortex methods.

A comparison of singular and desingular particle methods, and their vorticity updating equations is given by Winckelmans and Leonard [36]. An example of a modern approach is presented by Ploumhans [37], which uses a redistributed vortex particle method with a Particle- Strength-Exchange scheme and is solved with a parallel multipole treecode.

Despite their flexibility and grid-free nature, the easy computability of grid properties has led many researchers to use temporary grids for aspects of flow computation. Marshall and Grant [38] and Liu [39] use a grid to compute stretch. Others have used grids to compute vorticity diffusion [40]. A whole class of methods, VIC, are designed to compute the velocity on a grid.

A primary advantage of vortex particle methods is that they do not rely on structural properties of the tracked vorticity elements, and thus need not

track changes in their topology.

In particle methods, the strength of each computational point can be assigned in one of three ways: Hald [31] assigns the value of the vorticity contained in the surrounding blob, Beale and Majda [32] assign the value of the vorticity at the point times the volume of the blob, and schemes that assign a new value at every time step. Marshall and Grant [38] assign a value resulting from a global matrix calculation that insures that a divergence-free vorticity field results from the particle discretization. Actually, several schemes compute new "quadrature" weights at each step. Another is Strain [41]. The first implementation is credited to Beale [34].

Remeshing issues

The deterioration of the spatial accuracy of a vortex particle method is brought about by the separation of the individual vortices during their motion, causing a non-"smooth" representation of the vorticity field.

The primary methods to solve this problem are: recalculation of the quadrature weights at each time step [34, 41, 38], regridding/rezoning [42] (is this to new particles?), and global regridding to regularly-spaced particles (Cottet).

Beale and Majda CITE?? suggested and tested a rezoning technique whereupon the smoothness of the vorticity field was tested at every time step, and when it was determined to be too great, a temporary mesh was fit over the field, and the vorticity from all of the old particles was placed on the mesh. From these integer mesh points, new particles were created. The old ones were then removed. This is called remeshing, or particle redistribution.

Remeshing in its most common form means that at a given time, all particles are removed, and new particles are created at the center of cells on a regular grid. The particle strengths are set in order to recreate as accurately as possible the original vorticity distribution. This is done more frequently if the strain is high, often being done at every time step. Remeshing is used frequently in vortex particle methods [43].

An alternative version of this method [44] remeshes to the center of cells in an octree, instead of to a regular grid. The interpolation used [45] involves the 3 closest nodes in each of 3 directions (three 1-D filters convolved, as such, creates a non-spherical method), which exactly conserves total vorticity, linear impulse, and angular impulse, and somewhat conserves energy and enstrophy [44]. These routines typically have a minimum vorticity mag-

nitude for new particles to compensate for the diffusive nature of the Eulerian remeshing. This cutoff can influence the flow considerably (Gharakhani, private communication), though Cottet *et al* [46] claims that high-order frequent regridding causes no discernible numerical dissipation (even though the L_2 velocity error shows 10^{-3} error with a 4-th order kernel). The accuracy claims are somewhat refuted by Barba, Leonard, and Allen [47], who provide an error analysis of vortex blob methods.

Radial Basis Functions (RBF) were originally designed to allow approximation of a smooth field from a set of scattered data. RBFs are used by Barba, Leonard, and Allen [47] to remesh the vortex blobs in 2D simulations, where they show substantially lower errors for long-time calculations than even infrequent M4' remeshing. Unfortunately, to implement RBFs requires the solution of a system of N equations, where N is the number of particles in the system. It is possible, though, to use fast methods to solve this problem in $\mathcal{O}(NlogN)$ time. Another advantage is that RBFs allow consistent treatment of spatially-varying core sizes.

An alternative method [48] stores and tracks, for each particle, three vectors, originally orthogonal. As the simulation progresses, these vectors deform due to strain. If the length of any of the vectors increases past a certain value, the particle is split in two; if the length is below a certain value, the particle is merged with another close particle. The total vorticity is unchanged, but the kinetic energy and enstrophy are not explicitly preserved.

Mansfield [48] also found that the standard global remeshing to regular grid points, though maintaining total vorticity and enstrophy, caused artificial growth of the vorticity region due to the diffusive nature of the Eulerian remeshing (when the remeshing was kept to within 1% of maximum vorticity). There are "anti-diffusion" methods that have been created to prevent this, though, I believe, Cottet says that a sufficiently high-order remeshing filter prevents any undue diffusion of the vorticity region. Many authors, though [49, 50] simply remove particles with vorticities below a certain threshold.

Mansfield [48] shows that remeshing is necessary in flows with significant strain, despite the presence of any subgrid dissipation models. This should have been obvious.

Rossi [51] describes a method for merging Lagrangian particles, but only for methods that use Gaussian basis functions. The method could be converted for use with other basis functions. This procedure conserves the zeroth, first, and second moments of vorticity.

Remeshing is done by Cottet [9] and others [52] using a remapped grid with spatially-varying cell sizes. This allows finer resolution near shedding objects, and lower resolution in far-field wakes.

Ploumhans and Winckelmans [53] present a method for particle redistribution in 2D in the presence of general solid bodies. This redistribution is extended to 3D [37].

A detailed description of the remeshing of elliptical particles to regular axisymmetric particles is done in [54].

Chatelain and Leonard [55] present a method for particle redistribution to a face-centered cubic lattice, and show that it compares favorably with redistribution using M4' and "witch-hat" (M2) filters.

Eldredge [50] remeshes every few steps using the M4' kernel.

Note that in two dimensions, vortex methods can use particles for one of two uses: as free markers of vorticity, with no connectivity; or as connected markers defining a vortex sheet. The remeshing used is different for each type. In the vortex-sheet equivalent, early works [56] would add and delete particles along the line and reset vorticity values, some would even enforce a constant separation distance and fit cubic splines between particles. In the free particle method (no examples yet) vortexes are usually joined when they approach one another (or some more modern method is used).

Clearly, particle methods with remeshing are accurate and fast, but they are less useful to define sharp discontinuities.

Convergence

As detailed in Fishelov [57], the first proof for vortex methods was given by Hald and Del Prete [30] for the 2D case without viscosity (Euler's equations) in 1978. Beale [33] proved convergence of the 3D vortex method with stretch computed by differentiation of the smoothed kernel in 1986.

More information on the convergence history of particle vortex methods is contained in Haroldsen and Meiron [58]. In it, he cites Goodman, Hou, and Lowengrub [59] with the convergence of the point vortex method for the 2D Euler equations in 1990 and Hou and Lowengrub [60] for the convergence in 3D in 1990. Another convergence of PVM for 3D Euler is Cottet, Goodman, and Hou [61], and Hou, Lowengrub, and Krasny [62].

Perlman's [63] version gives the Hald [31] credit for establishing the convergence of 2D inviscid vortex methods. Beale and Majda extended those to higher order in two- and three- dimensions [29, 32].

The accuracy of particle vortex methods depends on many things, most notably the choice of cutoff function and core radius (sections 2.4.2), and the initialization of the vorticity distribution. These constitute the consistency error (smoothing error and discretization error). Also important is the stability.

Merriman [64] breaks the consistency error into moment error and discretization error and shows how simple discretization methods can lead to $\mathcal{O}(1)$ moment errors near boundaries. That work proposes and analyses different methods around that deficiency.

2.1.2 Vortex filament methods

The theorems of Helmholtz and Kelvin showed that tubes of vorticity retain their identity and move as material elements in constant-density, inviscid fluid. This led to the study of vortex lines and filament vortex methods. Batchelor [65] showed that a zero cross-section vortex line has infinite self-induced velocity anywhere its curvature is non-zero, obviating the need for regularization.

Vortex filament methods are characterized by sequences of material markers and space curves constructed to link them. Early filament methods used 1st order curves (straight segments), but higher-order splines have been implemented for either the Biot-Savart integration [66], the element remeshing, or both [67].

Vortex segments in a 3D vortex method were first used by Leonard [68, 12], probably. Chorin [28] used simpler, but still filament-connected method. Leonard [68] proposes topology change for vortex filaments in very close proximity.

Couët *et al* [22] was the first to couple a filament method with a 3D VIC solver, it used quadratic splines with 2-point quadrature points for the integration. Leonard's review paper [69] covers three-dimensional vortex methods using thin filaments with uniform core structure, and references many earlier works. Ashurst and Meiburg [66] extended the straight-segment vortex filament method by connecting adjacent nodes with cubic splines. The fixed vorticity distribution in the cross-section of the filament does not change due to local strain effects (but does change radius uniformly along the filament to maintain volume and conserve energy), and thus defines a small-scale resolution limit. Martin [70] applies this method to study the dynamics of a swirling jet. Knio and Ghoniem used a "thin tube" method, which is essen-

tially a vortex filament with a core size that varies due to stretch, to study the stability of vortex rings [71] and the roll-up and entrainment of a shear layer [72]. This essentially is the filament analogue of the core-spreading technique. Leonard [69] suggests that the core radius change uniformly along its length in order to make the filament volume constant. Local variations in the core radius, due to local differences in vortex stretching, can cause helical vortex lines and thus axial flows that would smooth these variations (Leonard's wording) [73].

Pothou [74] uses a vortex filament method to predict the acoustic field resulting from the impact of two vortex rings.

Knio and Klein [75, 76] show that using closely-spaced vortex particles to represent vortex lines introduces O(1) errors in the velocity. This improvement of the "thin-tube model" represents filaments more accurately.

Convergence

The convergence of the vortex filament method is presented in Greengard [77], the method for vortex stretching being a centered-difference operator along the filament elements, as done by Chorin [78] (really?).

Remeshing issues

Remeshing to maintain the quality of the discretization is almost always done by splitting any long filament elements in two. Its origins are likely in contour dynamics remeshing of surfaces in the 1D Vlasov equation by Berk and Roberts [79].

Occasionally, the new node in the material line is chosen with a higher-order function [67], or by using an FFT [80], but often it is merely the midpoint of the old segment [79, 78, 81]. Chorin [78] claims that a filament remeshing scheme more elaborate than midpoint/linear interpolation is unnecessary.

Leonard [68] did filament surgery, which joins filaments together in order to dissipate their circulations and reduce the element count.

A particular problem associated with filament methods appears when filaments are used to describe real vorticity distributions. A number of closely-packed filaments are commonly used to represent larger vortex structures [67, 71, 72, 80], but after a finite time, these structures thin in directions normal to the filament axes. Schemes have been created to maintain resolu-

tion in these directions, but they involve either breaking the vortex tubes [72] or adding an entire new tube between two existing tubes [82] Neither scheme provides a flexible, global method for filament remeshing that maintains an even element-to-area ratio.

Politis [83] uses vortex filaments to describe the motion of a shed vortex sheet in a wake. No remeshing is done at all, either along the filaments, or between filaments. The filaments are connected by rectilinear panels.

2.1.3 Vortex sheet methods

A vortex sheet can be either a topologically 1D front in a 2D simulation, or a topologically 2D front in a 3D simulation. A true vortex sheet method should be defined as one which maintains connectivity in $(D-1)$ space dimensions (both tangential to the sheet), whether the vorticity is represented by particles, filaments, or triangles/quadrilaterals. Methods that represent sheets with disconnected particles or filaments are not vortex sheet methods for purposes of this discussion.

A vortex sheet is commonly used in boundary element methods to describe the bound vorticity or dipole distribution of a solid or flexible (but still non-material) boundary within an irrotational flow. In these cases, the vortex sheet strengths of the boundary elements (usually triangles) is initially unknown, and must be solved for at every time step. These vortex sheets are related to the free (wake) vortex sheets described in this section. See section 2.3.2.

Vortex sheets had been used extensively in two-dimensional vortex methods, and have been represented as a collection of overlapping vortex filaments or vortex particles, but Agishtein and Migdal [84] was the first to demonstrate a three-dimensional vortex method based on vorticity discretization into sheets. This work used flat triangles.

Hou, Lowengrub, and Krasny [62] demonstrates the convergence of the point vortex method in describing the motion of vortex sheets. Lowengrub did this in his 1988 dissertation.

Knio and Ghoniem [72, 85] used triangular and quadrilateral elements to discretize the vorticity and scalar gradient in their simulations of doubly-periodic sheets. The velocity gradients ($\nabla \boldsymbol{u}$) are explicitly calculated using the gradient of the Biot-Savart equation. The displacements of the nodes are only used to check for low discretization accuracy. The vorticity is still discretized using overlapping thin tubes, like earlier three-dimensional im-

plementations, but they are co-located with the tri and quad scalar gradient elements, allowing local computation of the stretching term using the overlapping scalar gradient elements' node points. Of special note is that the triangular discretization used here did not constitute a complete and continuous surface, see figure 1(b) in the reference.

An improvement appears in Knio and Ghoniem [86], which uses only rectangular elements, but forces the vorticity to be exactly along an edge of the element. It still computes the stretching term as a filament (i.e. using the difference in lengths of the element along the vorticity direction). This work, though, includes the effects of small-Richardson number density gradients, which cannot be computed using the Lagrangian stretch method. Thus, "baroclinic splitting" is used, and both terms in the vorticity update equation (vortex stretching and baroclinic source term) are computed in a different manner. Ouch.

Brady *et al*[87] showed the use of triangulated vortex sheet methods, but no remeshing between sheets is done. The vortex sheet is C^1 smooth and defined on a two-dimensional parameter space, constraining the sheets to be topologically two-dimensional and simply connected at all times. The computations used a direct summation technique, with the influence of each triangle being calculated by integrating over a number of non-singular Gauss quadrature points. The elements are cubic Bèzier triangular patches, and as such have a continuous normal vector.

Lozano *et. al.* [88] used the equivalent of quadrilateral elements for the simulation of a strong density discontinuity.

Pozrikidis [89] used a C^1 continuous sheet discretization by using a network of quadratic curved triangles defined by six nodes. Quantities are approximated with quadratic basis functions. Still, tangential derivatives of surface functions and components of the normal vector are discontinuous across element edges. This work was the first to show the motion of a singular vortex sheet in three dimensions. There is mention that the first 3D sheet method without kernel regularization was Haroldsen and Meiron [58]. Beale [90] also presents method for solving singular integrals.

Many authors use discretized vortex sheet elements in the computation of 2D and 3D boundary-layer flows to track vorticity diffusion into the fluid [91, 92, 93]. But these elements are bound to a solid surface, and do not directly shed into the flow.

An interesting adaptation of vortex sheets appears in Summers and Chorin [94], which presents a method of creating impulse at solid boundaries and

converting them to vortex loops as they leave the numerical boundary layer. These loops are intrinsically divergence-free, and are equivalent to vortex loop panels set free into the flow. See §2.1.6.

It is important to note that vortex sheet elements are assigned their strength differently that particle methods.

Convergence

Caflisch and Lowengrub [95] present a proof for convergence of the vortex method for vortex sheets.

Remeshing issues

In a 2-D particle method, some connectivity can be created and enforced. Tryggvasson [96] introduces a method to maintain the resolution of the connected "sheet" by automatic insertion of particles as the sheet is stretched. The vorticity of these particles is naturally modified, per the vorticity transport equation. I believe Krasny did this same thing in 1986. Similarly, Dritschel [97] introduced a similar method for two-dimensional contour surgery. Another 2D sheet method was recently used by Kim *et al* [98], but used *global* remeshing instead of Krasny's point insertion technique (cite?). From these humble beginnings...

Knio and Ghoniem [72, 86] used a three-dimensional vortex sheet method with rectangular transport elements containing information of the vorticity, scalar value and scalar gradient. These rectangular elements were remeshed by splitting the element in two in areas where high strain reduced the resolution below the core smoothing radius. This remeshing is done separately in both directions. The earlier work [72] also investigated triangles, but the remeshing did not maintain the connectivity of the sheet, it created holes in the scalar surface, but since the vorticity was discretized essentially as filaments, did not affect it. The problem with this method is that if the primary extensional strain axis is oriented diagonal to the rectangular mesh, the creation of a large number of very thin elements is unavoidable. Not even the remerging of nodes and elements will maintain the core overlap. Arbitrary triangular meshes with merging avoid this problem entirely. Evidence of this appears in Table 3 in [86], which shows the number of elements increasing super-linearly with the area of the discretized surface.

Brady *et al* [87] did remeshing within a sheet by maintaining element

quality in areas of high curvature. This does not allow long-time runs, as not only does the frontal area increase exponentially, but the sheets tend to create areas of high curvature. Triangular elements were used. The method maintained the circulation on the elements as constant and, thus, had no problem with vorticity divergence.

Pozrikidis [89] does no remeshing. There is mention that spectrum truncation may be necessary for vortex sheet simulations involving surface tension or density discontinuities, but that an effective method of smoothing the positions of marker points defining a triangulation has not been developed. Kwak and Pozrikidis [99] present a simple method for regridding directly in physical space, as the current method does.

There is an additional problem associated with vortex sheets, and that is in the representation of sheets of finite thickness with a number of overlapping vortex sheets. In the presence of roll-up, the sheets will separate in the direction normal to their surface, losing resolution and compromising the accuracy of the method. In numerical experiments by this author, even very-closely-spaced sheets ultimately separate. Look for research on the desingularization of vortex sheet methods by the overlaying of several singular vortex sheets. See figure 14d of [86]. Baker *et al* show the beginnings of this behavior in a three-fluid system [100].

Extensive discussion of remeshing of 3D triangulated meshes is contained in Tryggvason *et al* [101] and Glimm *et al* [102, 103]. These methods require logical connections between triangular elements. Newer methods have been created that relax that requirement, such as Torres and Brackbill [104] (require quintic splines to achieve accurate curvatures, but can actually determine surface element area from an unconnected set of points), Shin [105] (a level-set method with marching cubes stuck over it—the marching cubes triangles are not linked, though they share endpoints), and level set methods (section 2.1.5).

Wang and Khoo [106] use a method called the Elastic Mesh Technique (EMT) to track an interface between water and air. It uses a relaxation method to move mesh nodes, providing higher-quality meshes than without. It does not allow for element insertion.

Aulisa *et al* [107] presents a method for maintaining enclosed volume in a combined Eulerian-Lagrangian front-tracking method in 3D.

The problems with all of these methods are twofold: surface detail below Δx in size is lost, usually without any subgrid effects; and because the surface is entirely remade, no element-wise information can be retained (vortex sheet

strength, moments of scalar gradient).

No one, to my knowledge, has attempted inter-sheet remeshing in a vortex method. One should look to transport element methods or to the computer graphics literature for examples of this kind of remeshing.

Smoothing issues

In order to prevent the growth of unwanted small scales, some authors have taken to applying smoothing over the field of connected nodes. This is usually only done with segment (in 2D) and filament (in 3D) methods. Kim [98] filters the global curve in frequency-space. Tryggvason did much the same in earlier 2D methods.

2.1.4 Vortex volume methods

Discretization of vorticity can be at one higher dimension, still: volumes. A Lagrangian mesh can be defined by material points in space and volume elements defined by their connection to these material points. A triangulated vortex method in two dimensions was first introduced in Russo and Strain [108]. Strain [41] claims that it is difficult to make a triangulated vortex method (2D version) that is more than second-order in space, though a later generalization [109] remedies that.

An axisymmetric version exists by Carley [110]. Huyer and Grant [111] have demonstrated two-dimensional volume discretizations using Delaunay triangulations. Grant and Marshall [112, 113] have devised a method that used tetrahedra to discretize the vorticity. The tetrahedra are created via Delaunay triangulation over a series of particles. Vorticities are assumed to vary linearly within the tetrahedra. Influences are calculated using treecode/FMM, with local interactions using Gaussian quadratures or integrals over the volumes of the elements.

A two-dimensional vortex volume method uses a triangular mesh to represent areas of scalar-valued vorticity. To maintain discretization accuracy, the distorting triangles need to be remeshed onto a new set of less-distorted triangles. One possible procedure for conservatively remeshing the vorticity is presented by Ramshaw [114], though it is unknown whether or not is has been used in the context of vortex methods.

2.1.5 Level Set method

The Level Set Method introduced by Osher and Sethian [115] in 1988. A summary of recent level-set research appears in [116, 117]. This is an example of a "front capturing" technique, which is different from a "front tracking" technique in that the latter maintains an explicit representation of the interface. Level sets are amenable to grid solvers, hence their popularity.

This method can be used to capture sharp interfaces between fluids, though it requires a large number of tracked particles, and automatically coalesces fronts when details are smaller that a grid cell. It was first used for flow calculations in [118]. In that work, the method appears diffusive and less-capable of tracking detail than a pure Lagrangian method.

Chang *et al* [119] used a level set method in 2D to compute the motion of interfaces tracking large density jumps. The solution method is pure Euler, though. Zhao [120] uses a variational level set approach for multiphase motion.

Strain [121] presents a quadtree level set method for moving interfaces that uses a semi-Lagrangian advection technique.

Herrman [122] uses a level-set/vortex sheet method to study two-phase flows in two dimensions to study the primary atomization in turbulent environments. The method has no viscous diffusion (used the Euler equations), but accounts for surface tension.

The Particle-Level Set method [123] tracks Lagrangian particles, but uses them only to recreate the grid-resolution-limited front for the Euler calculation.

Disadvantages of the level-set scheme are that it relies on high-order schemes for the advection term (being an Eulerian approach), and, for the vortex method extension, requires special treatment of the vortex sheet strength transport equation. In addition, the resulting velocity field for incompressible flow calculations typically contains non-zero divergence, and thus a separate calculation must be made to make the results divergence-free. Its advantage is that the curvature and normal can be calculated on the grid, allowing for smoother interpolation of curvature than a C^1 continuous sheet (piecewise flat triangulated mesh).

Only recently [124] have level set methods been used to track material quantities on propagating interfaces. Is this in a Lagrangian sense yet?

2.1.6 Magnet/impulse elements

Lundgren [125] proposed the use of spiral vortex cores (a preferred state of stable vorticity) in the study of fine-scale turbulence. These cores are clearly the cause of the cauliflower look of rapidly-convecting cloud boundaries, and can even be seen in Javier's PLIF images of buoyant turbulent jets.

Buttke [126] reformulated vortex methods in terms of impulse elements, as they naturally preserve the divergence-free character of the vorticity field.

This type of element must be related to the vortex dipole in R. Cortez's thesis (UC-Berkeley, 1995). This element preserves all invariants of the three-d Euler equations, as well as maintaining the solenoidality of the velocity and vorticity fields. The singularity of the velocity kernel is of one order higher than the standard vortex monopole, and this results in not only stability restrictions, but requires a solver that can handle r^{-3} velocity influences (note that velocity gradients in vortex monopole fields in 3D decay like r^{-3}, as the velocity itself decays as r^{-2}).

Called a "magnet" method in [94], these elements are conceptually equivalent to a vortex loop. The magnitude of the impulse of a magnet element is equal to the product of the vortex loop's circulation and area. Summers and Chorin [94] shows how to create impulse and vorticity on a solid surface.

Zabusky [127] calls coherent vortex structures that move under their own self-induction "vortex projectiles." Examples are a pair of point vorticies in 2D or Hill's spherical vortex in axisymmetric coordinates or 3D.

Cortez [128, 129] shows that impulse blobs are roughly equivalent to vortex blobs in some flows, but cautions their use for representing vortex sheets under extreme strain.

2.1.7 Semi-Lagrangian particles

As an option, one could use a semi-Lagrangian forward time-stepping scheme where the state at the new time step is determined using a Method of Characteristics—walking backwards in time along the local velocity vector and interpolating the value at the resulting position. This method originated with work from Wiin-Neilsen [130] in 1959 and Sawyer [131] in 1963. Good descriptions appear in Robert [132] and a history appears in Bates [133]. Jos Stam's "stable fluids" [134] added a velocity correction step to account for divergence errors in the velocity advection stage.

Advecting *vorticity* instead of velocity would produce a method that is

much less dissipative, but slightly more costly. It would be no more stable than the velocity formulation as published, but it will be less diffusive. This was done by early authors [131] in two dimensions, but not in three dimensions until Malevsky [135].

Also notable is a Semi-Lagrangian method for contour advection, called the "Contour Advection Semi-Lagrangian" (CASL) algorithm by Dritschel [136] and used in [137]. Alternative Semi-Lagrangian methods for surface advection can use level sets [121].

2.1.8 Pure Eulerian

Though not vortex methods *per se*, Eulerian (grid-based) solutions of the velocity-vorticity or velocity-streamfunction equations relate in many ways to their Lagrangian counterparts. Many vortex methods have borrowed algorithms and methods from Eulerian implementations.

Some examples of grid-based vorticity methods are [138, 139, 140, 141].

2.1.9 Combinations

Several authors see merit in combining two of the above methods in their calculations. For example, Chorin [142] first combined sheet with particles in 2D, later done by [143]. Bernard combined sheets with filaments for his late-90's work [144].

2.2 Solution methods for the Biot-Savart equation

2.2.1 Direct integration

A vortex method is a computational method in which the flow is represented by a collection of Lagrangian particles of vorticity moving under the self-influence of one another. This motion is quantified by the Biot-Savart law (2.1), which determines the velocity at a point in space given a complete definition of the vorticity field.

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{4\pi}\int \frac{\boldsymbol{\omega}(\boldsymbol{x}',t)\times(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,d\boldsymbol{x}' \tag{2.1}$$

Using this formula, a vortex particle's velocity can be computed from the vorticity and location of every other particle. Each particle is then advected according to its local velocity, and has its vorticity modified to account for vortex stretching and viscous effects.

It is obvious that if the Biot-Savart law were used to calculate the velocity of each of the N particles in a simulation, then the calculation of a single time step would involve $\mathcal{O}(N^2)$ evaluations. This is clearly inappropriate for large values of N, as would be necessary for simulations of three-dimensional turbulence.

One method to reduce the $\mathcal{O}(N^2)$ computational effort is to use a long-range cutoff—to ignore the influence of any vorticity that is beyond a threshold distance. This was used by Chorin [78], but is more frequently used for Lennard-Jones forces (which vary as r^{-6} and r^{-12}) in molecular simulations [145].

Sometimes called "direct evaluation" or the "direct method", this method is straightforward to implement, but slow to run. Implementing the method on parallel computers [146, 147, 92] is straightforward, but only provides a linear speedup dependent on the number of processors involved.

Lozano *et al* [88] applied this method to solve for the evolution of a vortex sheet carrying a large density jump. A δ parameter was added to desingularize the Biot-Savart kernel. The kernel was integrated over the entire computational surface.

Low [148] first suggested that vorticity can be discretized as blobs onto material particles and may travel with values intact. The first vortex simulations were done by Rosenhead [16] in two dimensions with 12 singular point vorticies. Those same calculations were repeated by Birkhoff and Fisher [17]. Those calculations showed that without regularization, a point vortex method cannot be used to approximate a vortex sheet. See discussion in [149].

Regularization in vortex methods

The need for regularization was driven by the discovery of curvature singularity formation in vortex sheets in finite time in 2D [150, 151], 3D [152, 153, 154], and in contour dynamics [155] in 2D [156]. Kudela [157] goes so far as to state

> From the practical point of view, the emergence of a singularity is physically unacceptable and shows inadequacy of the mathe-

matical model in describing the problem. So it is reasonable to include certain physical mechanisms like diffusion, interfacial tension or the finite thickness of the interfacial transition region to regularize the problem.

Regularization, as opposed to diffusive viscosity, can be provided by a number of means, from length scale cutoffs [15, 19, 26], in direct or treecode/FMM methods, to the grid regularization provided by hybrid Eulerian-Lagrangian schemes such as VIC and Level Set [118, 122]. Kudela [157] points out the viscous character of the δ^2 regularization [26].

Nitsche [158] says "comparisons with solutions of the Navier-Stokes equations [159] and with experimental measurements [160] show that the [vortex blob] method approximates viscous flow well for sufficiently small values of the artificial smoothing parameter and viscosity."

Luchini and Tognaccini [161] compare high-resolution two-dimensional simulations of regularized Euler flow and high-Re viscous flow and find very few differences.

Keep in mind that the Biot-Savart kernel exhibits a $1/r^2$ singularity, thus a vortex sheet must have a continuous normal vector in order for the principal value of the integral to exist.

Dissipation is an intrinsically different procedure than regularization, though tests [161] have shown that the two behave very similarly in certain cases. This is because regularization in a vortex method limits the magnitude of the velocity gradient, which in turn prevents the stretching term from creating the required new vorticity.

2.2.2 Vortex-In-Cell (VIC)

A vortex-in-cell scheme can be used to determine the velocity *field*, instead of the individual element velocities, as direct integration of the Biot-Savart equation would produce. VIC is a pure PM (particle-mesh) algorithm, and is an extension of the Cloud-in-cell (CIC) algorithm. Birdsall and Fuss [162] introduced the Cloud-in-cell method for plasma particle flows in 1969. The governing equations for these flows are very similar to those of the streamfunction equations, as both rely heavily on potential theory. The first particle-in-cell method (Harlow [163]) was for hydrodynamic problems, used the zero-size-particle and nearest-grid-point method (ZSP-NGP) and did not use a Poisson equation.

First, the elements' vorticities are placed onto a temporary grid. From this vorticity field, the velocity field is solved for in one of two manners:

$$\nabla^2 \boldsymbol{\psi} = -\boldsymbol{\omega},\ \boldsymbol{u} = \nabla \times \boldsymbol{\psi} \tag{2.2}$$

or

$$\nabla^2 \boldsymbol{u} = -\nabla \times \boldsymbol{\omega}. \tag{2.3}$$

In two dimensions, these equations can be written

$$\nabla^2 \psi = -\omega\, \hat{e}_z,\ \boldsymbol{u} = \nabla \times \psi \tag{2.4}$$

and

$$\nabla^2 \boldsymbol{u} = -\nabla \times \omega\, \hat{e}_z. \tag{2.5}$$

The streamfunction formulation requires one solution to the Poisson equation for the 2D case (2.4), and three for the 3D case (2.2). This formulation is also discussed in [140], and in 3D requires that the vector potential, vorticity, and velocity are all divergence-free. More discussion on the need for divergence-free vorticity appears in [164]. A problem with the streamfunction approach in three dimensions is the determination of proper boundary conditions. Wong and Reizes [165] propose a solution in this regard. This formulation originates with Helmholtz's Decomposition Theorem.

The vorticity formulation requires two solutions of Poisson's equation for two-dimensional flow (2.5), and three for three-dimensional flow (2.3). An advantage is that the only requirement is that the velocity be divergence-free. There is no mathematical constraint on the vorticity, save our insistence on its divergence-free physical character. (Is this really true?)

The Poisson equation, being a special case of the Laplace equation (which is itself a special case of the Helmholtz differential equation), is a separable partial differential equation, and can take advantage of fast solvers that use FFTs (FISHPAK) or multigrid methods (MUDPACK). Leonard [12] goes into further detail on the relation of this solution to Fourier methods.

Christiansen [166, 167] first applied the VIC method to fluid flow problems in 1973, using a code developed in 1970 [168], for the simulation of the Kelvin-Helmholtz instability in two dimensions. He used the streamfunction approach (2.4).

At the same time, Hirasaki and Hellums [169] presented a solution method for the three-dimensional vector potential requiring three separate Poisson equations with mixed Dirichlet and Neuman boundary conditions.

Couët [22] first used a three-dimensional VIC method, using vortex filaments to study the evolution of vortex rings. They noted two sources of azimuthal velocity unevenness: one caused by the interpolation kernel itself (causing small-scale perturbations), and the other caused by the doubly-periodic placement of image vortex rings (causing a larger $k = 4$ mode perturbation).

Zawadski and Aref [170] present a 3-D VIC method using vortex sticks (disconnected particles) to simulate offset vortex ring collisions.

Ould-Salihi [11] compares a VIC method and a finite-difference method in his discussion of combined particle-grid methods. Savoie, Gagnon, and Mercadier [171] used a random-walk VIC method to compute the starting flow behind a two-dimensional step. Tryggvason [149] compares VIC methods to vortex blob methods by studying the Kelvin-Helmholtz instability in two dimensions. Abdolhosseini [172] use a two-dimensional VIC method to study the growth of turbulence in a shear flow. Cottet [10] compares a VIC method to a spectral method to calculate homogeneous turbulence.

In addition to the errors present in direct calculation, this hybrid approach introduces several new error terms. Due to the repeated interpolation of values between the particles and the grid, the error of the interpolation method must be studied. Additionally, differencing errors from the Poisson solution must be accounted for.

Meng and Thomson [56] state that VIC calculations are inherently unstable (true for all vortex methods, I would think), and this is corrected for by filtering in wavespace and damping the high wavenumber modes. Does this correct for the non-sphericality of the filtering kernel used?

The VIC method can be improved with a method of local corrections [173], whereby the effects of vortexes within a distance of $\mathcal{O}(h)$ are calculated using direct integration, while the effects of all other particles are calculated with VIC. This is an extension of the particle-particle/particle-mesh (PPPM) originally of Hockney, Goel, and Eastwood [174], 1974. Walther and Morgenthal use an improved PPPM [175] in their 2D immersed boundary VIC simulations [176].

The literature contains many examples of two-dimensional [172, 177, 46, 137] and three-dimensional [46, 178, 49] VIC methods.

2.2.3 Treecode/Fast Multipole Method (FMM)

Appel's method [179] and the Barnes-Hut method [180] both hierarchically subdivide a set of computational elements and approximate their strengths in order to speed the Biot-Savart velocity summation over the given set. The latter is also commonly called a treecode. While the original work used only first order multipole expansions (monopoles), methods using higher-order expansions are still referred to by this name.

The Fast Multipole Method (FMM, Greengard-Rokhlin) [181] is an extension of the Barnes-Hut treecode that includes calculating local expansions to reduce the order of the computations. While theoretically $\mathcal{O}(N)$, no implementation of FMM on desingularized particles achieves that performance. Researchers have achieved speeds of $\mathcal{O}(NlogN)$ or even $\mathcal{O}(N^{1.1})$ [44] (but it looks more like 1.2) using these methods. Dehnen [182] claims $\mathcal{O}(N)$ speed, but only for singular particles.

Anderson [183] presents a fast method similar to FMM but which uses Poisson integrals instead of multipole moments. Performance is compared to FMM in [184], appendix b.

In an interesting observation by Nordmark [42], the computational labor for FMM is limited to $\mathcal{O}(N^{1.5})$ when high- or infinite-order cutoff functions are used. This is because the cutoff parameter δ must be proportional to $\sqrt{h}$ [185], where h is the grid size for regridding, in order to maintain high accuracy for long-time integrations. When δ is proportional to h, FMM can achieve its theoretical optimum order. Strain [109] uses a new error bound to construct a 2D vortex method that shrinks the need for large smoothing radius.

The Barnes-Hut/treecode method is more easily parallelizable, and may have been done in a scalably fashion first for gravitational problems by Salmon [186]. Since then, several authors have presented parallel treecodes [187]. Bernard [144] uses the full G&R FMM code on parallel machines.

Treecodes [188] have been used to calculate velocities in Eulerian-frame Navier-Stokes solvers in vorticity variables.

FMM/treecodes (particle-particle methods) have also been used for molecular simulations [189, 145], but Particle-mesh methods with local corrections (particle-particle particle-mesh) may be equally or better suited.

Strickland *et al* [190] present a treecode that solves for both velocity and radiation in three dimensions. The authors mention that treecodes can be used to solve scattering problems, too.

The literature contains many examples of two-dimensional treecode [188], three-dimensional treecode [190, 37, 191], two-dimensional FMM [41, 109, 192], and three-dimensional FMM [144, 43, 192] implementations.

2.2.4 Other methods

Contour dynamics, which have no three-dimensional counterpart, are effective in computing flows with bounded areas of constant vorticity [155, 193]. This avoids the problem of point vortex methods by changing the order of the singularity in the integrand from x^{-1} to $\log x$. The origin of this method is the Water-Bag model of Berk and Roberts [79].

A related method [194, 195], possibly inspired by the results in [193] uses overlaid elliptical vortex patches to describe the motion of separated patches of vorticity in 2D flows.

2.3 Corrections due to boundaries

Talk about the additive nature of the velocity field, the superposition of potential flow elements, and the solution of panel methods [196] and their parent Boundary Element Methods.

Two types of boundary must be defined here: one type of boundary is the grid boundary used in hybrid Eulerian-Lagrangian solvers such as VIC; the other is a boundary surface between the fluid domain and a fluid-impenetrable object, whether the flow be internal or external to the surface. The second type of boundary will be discussed here, but there must be some overlap of the descriptions due to their similarity in some cases (inviscid slip wall boundaries in VIC).

This section should really be composed of the following: computational volume boundaries for VIC methods (Poisson equation solvers), boundary element method solvers for both free-space solvers (internal and external) (direct, treecode, FMM) and VIC (internal boundaries only).

The first boundary treatment in vortex methods was the *method of images*, a technique created to simulate the effects of a solid plane or sphere in a two-dimensional flow. Each singular vortex has an image vortex placed on the other side of the plane, or the interior of the circle, which participates in the velocity integration. The effect of these image vortexes is that of an

inviscid plane or sphere in the flowfield. This has been repeated in three dimensions [94].

Another wall-bounded flow method is used by Gharakhani and Ghoniem [93]. Summers and Chorin [94] use a hybrid vortex/magnet field to simulate the flow over a sphere.

Flows can sometimes be computed with periodic boundary conditions. This is possible with direct solvers with the use of image vorticies, and with VIC solvers, because of the flexibility of available Poisson solvers.

One problem with defining two types of boundaries, as we do here, is that for simply-connected domains, the mesh can be contorted to fit the internal boundaries. In the case of flow over a sphere [37], even though the authors use FMM, they still remap a grid spherically over the solid object and remesh particles on it.

2.3.1 Domain boundaries

The boundary of the computational domain is treated differently based on the velocity solution method. FMM and treecodes must use open boundary conditions exclusively, unless an Ewald-summation-style method is used to allow periodic boundaries. Wall boundaries, or any internal boundaries, typically require a full BEM solution.

Wall boundaries

VIC methods, due to their Laplace solver, can incorporate Dirichlet (value given on the boundary), Neumann (derivative given on the boundary), or periodic boundaries easily. Wu [197] discusses these types of boundary conditions for wall and free surface flows.

In addition, conformal mapping can be used to create a regular grid in domains that are not perfectly rectangular. Malarkey and Davies [198] investigate whether Routh's Correction is necessary in this case.

If an internal boundary is regular (spherical or rectangular), the computational domain for an Eulerian-type calculation can be redrawn to conform to the internal boundary. Most notably, Cottet has done VIC calculations of the flow around a sphere and cylinder [199, 49] using this kind of body-fitted grid.

This section covers not only methods for which one face/side of the computational volume is defined to be a solid wall, but also for methods which

map or contort a regular grid such that the same condition exists [37, 200]. Conformal boundaries are even used in methods where the boundary is moving, such as a wavy moving wall [143].

Tryggvason *et al* [139] compute flow inside a 2D box with the top assumed to be a free surface with variable surface tension. The work uses finite differences and a u-ψ formulation. Because the free surface is treated as a boundary of the fluid domain, it is included in this category.

Open boundaries

Baker [201] computes the boundary conditions for a VIC method using the streamfunction approach by grouping all of the point vorticies into local centroids and calculating the velocity at the open boundaries by direct summation of the effects of these centroids. Then, the values for the boundary streamfunction are determined by numerically integrating the around the boundary. It is even suggested that, with proper choice of the number of local centroids and number of boundary evaluation points, this method is faster than setting the boundaries as periodic in the Poisson problem. Liu and Doorly [39] computes the velocity on the boundary of their VIC domain by direct Biot-Savart integration over the cell vorticity values.

Brecht and Ferrante [202] refer to their development of free-space boundary conditions for a 3D VIC method using the streamfunction approach. A tridiagonal system is solved at each boundary. Though, in [203], they use periodic boundaries after stating that it changes the answer by only a few percent.

Gharakhani [204, 93] proposes a new type of outlet/open boundary condition for vortex methods which use the vorticity-streamfunction equations.

E and Liu [140] describe a separate 2D calculation at the outlet plane in order to solve for the outlet velocities.

Comini *et al* [205] employ "advective derivative conditions" at outflow boundaries for two-dimensional flow in ψ-ω vorticity coordinates.

Sohankar *et al* [206] study the effects of a convective boundary condition (the Sommerfeld BC) on the local and global flow over a square cylinder with a finite-difference method.

Gharakhani, in private conversations, suggests using FMM to solve for the boundary conditions (velocity) on all open walls, on a grid-wise basis. These values can be fed into a VIC solver as Dirichlet boundary conditions. The accuracy of the FMM summation would scale the velocity divergence

error.

Despite more advanced methods, modern approaches have skirted the difficulty in their own ways. An inviscid slip boundary is used in the VIC method in [11] to simulate an open boundary.

Periodic boundaries

In direct summation, treecode, and FMM implementations, which are most commonly run for free-space problems, one can implement periodic boundary conditions by either including in the computation the effects of a finite number of copies of the vorticity field across each periodic boundary [78, 66, 67, 88, 207], by a complicated summation and estimation of the error term [72] (obvious asymmetries remain in the results, see the y-plane images from [72, 86]), or by a method known as Ewald summation [58, 89]. Brady *et al* [87] discusses the singly periodic vortex sheet solution, while Pozrikidis [89] does the same for doubly-periodic free-space Green's functions.

In VIC methods, periodic boundaries are often supported in the Poisson solver itself, obviating the need for any special treatment [203], save the proper discretization of elements across the boundary [11]. This author has still seen asymmetries in flow simulations, usually after considerable flow development.

Symmetry boundaries

Baker [201] states that a plane of symmetry, when used as a boundary in a streamfunction-VIC calculation, can be defined by assigning a constant value to the streamfunction on that boundary. This is because a plane of symmetry in a 2D flow is also a streamline.

2.3.2 Internal boundaries

In this section, we will look at methods to solve for flows around solid objects that are at least somewhat within the computational domain, i.e. objects that displace fluid in the domain, as opposed to simply boundaries of the domain. The treatment of these boundaries depends on the type of solver used and less on the vorticity discretization.

An immersed boundary technique is given in Peskin [208] for the Navier-Stokes equations. Fogelson [209] extends a method by LeVeque and Li [210]

for discretizing the Laplacian with Neumann (derivative) boundary conditions for immersed interfaces in Cartesian grid solvers. The authors improve upon the first-order method (one-sided differences for the normal derivatives and centered differences for the interior Laplacian) by imposing a compact (3x3x3) stencil for the derivatives, making a second-order method. Deng *et al* [211] describes a method for solving 3D elliptical equations with immersed interfaces. Gilmanov *et al* [212] present a second-order technique for enforcing boundary conditions on immersed arbitrary triangle meshes in regular grids. Finally, Marella [213] discusses cartesian grid methods with immersed interfaces (either fluid-solid or fluid-fluid).

There seem to be many Eulerian methods designed to solve immersed boundary problems [214, 215, 216].

Boundary Element Method (BEM)

To compute flows around objects, one can use any number of methods. A boundary element method can be used to satisfy the no-through-flow or no-slip conditions for internal or external boundaries [93]. The BEM solution can involve direct calculation, and LU decomposition for inversion of the influence matrix [204, 147], or involve multipole-accelerated methods [44]. The simplest BEM is identical to the early panel methods.

Boundary integral techniques can do a number of things for a vortex method. In the inviscid case, it can solve for the distribution of source strength over an object in order to determine the irrotational portion of the flow in the Helmholtz decomposition [217]. From this strength follows the pressure, and then the forces. One could convert the pressure into a slip velocity by numerically integrating over the surface using the following equation

$$\delta \boldsymbol{u}_s = \Big(-\frac{1}{\rho} \nabla_s p + \nu \left. \frac{\partial^2 \boldsymbol{u}}{\partial \eta^2} \right|_s \Big) \delta t \tag{2.6}$$

where the flux of vorticity is obviously related to the second component.

Alternatively, a BEM could solve for the surface vorticity distribution by setting a no-slip boundary condition, and remove the requirement for a surface integral to be solved to determine the vorticity flux. This method requires solving a matrix that is twice as large in each direction, as the surface vorticity has two unknowns per panel. Spalart ("Vortex methods for separated flows," Von Karman Inst. for Fluid Mech., Lecture Series 1988-05, 1988) claims that in 2D these two cases are equal.

Clarke and Tutty [218] present a 2D BEM for vortex methods that uses elements that are 2nd order in space and 1st order in vorticity (linearly-varying vorticity over a curved element).

Wu and Wu [219] wrote complete equations for the vorticity, vorticity flux, compression/expansion, and force on a solid-fluid interface.

Marshall and Grant [38] use a combination of source and vortex panels.

Modern vortex codes use this boundary integral equation method, namely Ploumhans *et al* [37]. They also modified a method to compute the force on a surface to use only the velocity on the surface. Other modern VM-BEM research appears in [52].

Beale [220] solves elliptical boundary integral problems by determining the strength of a double-layer potential and uses rectangular grids in overlapping coordinate systems.

Most importantly, strong fluid density interfaces require the use of an elliptical boundary integral solution to determine the strengths on the interface. Really? See Baker and Beale [221]. Wang and Khoo [106] use a BEM to determine the motion of an multi-fluid interface for simulations of bubbles underwater.

Iterative methods

Instead of solving a dense matrix equation, one is able to obtain a solution by solving a series of elliptic equations for the surface strength of the solid (or free-surface) boundaries. This is also a BEM, but is constructed and analyzed in a different way.

Cottet [199] presents a method for satisfying the no-slip boundary condition on an arbitrary object in a rectangular grid. It consists of solving for a function g such that solving

$$\nabla^2 \phi = g \tag{2.7}$$

yields no through-flow on the boundary:

$$\frac{\partial \phi}{\partial \boldsymbol{n}} = -(\nabla \times \psi) \cdot \boldsymbol{n} \tag{2.8}$$

where ψ is the vector potential. Also addressed are schemes for shedding proper vorticity into the flow from these boundaries.

Udaykumar *et al* [222] present a method for solving the Navier-Stokes equations on a fixed rectangular grid in 2D around immersed boundaries, which requires a Poisson solution to the pressure equation.

Walther and Morgenthal [176] proposes another immersed boundary technique for VIC methods, but strictly in 2D.

A 3D immersed boundary method is used in VIC calculations by Cottet and Poncet [49].

Albuquerque and Cottet [223] present a method that couples a finite difference and integral formulas to create an iterative method for solving problems such as this.

2.3.3 Fluid-structure interaction

Fluid-structure interaction (FSI) requires coupling between a fluid solver and a structures solver. Many researchers have presented results with one-way, or no coupling. Only few have attempted full two-way coupling using vortex methods as the fluid solver [224].

2.4 Other topics

There exist several peripheral topics germane to vortex methods research that do not fit nicely into any of the headings above. These include particle-grid operators and the generalized Helmholtz decomposition. They shall be described in detail in this section.

2.4.1 Time integration and discretization

Most authors use 1st or 2nd order forward integration methods, and often higher order methods for advection are not similarly used for updating the particle strengths [184]. This is called flux-splitting. Some tests revealed that 1st order time stepping accuracy was not acceptable [44]. This is likely due to time integration errors in Lagrangian trajectories that may contribute to an inaccurate vorticity representation. Many authors still use first-order stepping for their diffusion routines. A 2nd order method from the Runge-Kutta family is used frequently [81], and so is the classic 4th order Runge-Kutta [98].

Different authors propose different maximum time steps. Brady *et al* [87] uses a 2nd order Runge-Kutta method with a time step limit based on a maximum element strain criterium. Other authors have used $\| \omega \|_{max}^{-1}$.

In simulations with large density jumps (section 3.5.2) the acceleration must be computed. Brecht and Ferrante [203] describe a leapfrog method that stably computes the acceleration term. This is dealt with more in the corresponding section.

Fernandez [187] uses temporal grid adaptivity (multiple substeps for fast-moving particle) for vortex filament simulations.

2.4.2 Particle-grid operators

IMPORTANT: note the difference between cutoff functions for pure vortex methods and particle-grid operators for combined Lagrangian-Eulerian methods!

Any particle-grid method relies heavily on the transfer of quantities from the particles to the grid and back. Most often, the interpolation kernels used are tensor products of one-dimensional kernels. A class of these are created by successive convolutions of the top-hat filter [184]. A straightforward analysis of these operators is given in [11].

To desingularize the Biot-Savart kernel, point vorticity values are convolved with a smoothing function that aims to replicate as many moments of the delta function but have a finite area/volume. The spatial order of accuracy of a vortex method is determined by the choice of this smoothing function.

Hald [31] presented several cutoff functions which emit second order accuracy. Beale and Majda [225] present a class of infinitely-differentiable functions which can provide high-order accuracy. Perlman [63] concludes that higher-order kernels improve accuracy only if the flow is smooth enough, and that the kernel support needs to increase as the kernel order increases.

Tryggvason [149] showed similar results between VIC methods and desingularized particle methods when the blob size and the mesh size are roughly equal.

Dissipation is an intrinsically different procedure than regularization, though tests [161] have shown that the two behave very similarly in certain cases. This is because regularization in a vortex method limits the magnitude of the velocity gradient, which in turn prevents the stretching term from creating the required new vorticity.

Interpolation filters are important because they govern both placement of values from Lagrangian element to a field, and the transfer of field quantities back to the elements. Higher-order methods for this, a general aspect of

hybrid algorithms, is presented in Walther and Koumoutsakos [226], who use VIC and PSE to include large number of solid particles into a flow with 2-way coupling.

In the current research, it is seen that anisotropic kernels (those created simply with tensor products of one-dimensional kernels) create small-scale structures that *do* influence the larger scales.

In placing vorticity onto a grid (or otherwise interpolating it), the discretization technique drives the formula, with particle techniques summing over $\boldsymbol{\omega}_i \, dV_i$ (vorticity times volume), filament methods summing over $\Gamma_i \, d\boldsymbol{X}_i$ (circulation times material segment vector), and sheet methods integrating $\boldsymbol{\gamma}_i \, dA_i$ (sheet strength times area).

Particle-In-Cell

The earliest Lagrangian-Eulerian methods [163] tied particles to grid values based simply on which cell each particle was in. Particles were considered to have zero volume and pass instantly from one cell to another. This is equivalent to a nearest-neighbor interpolation method. This is also called zero-size-particle and nearest-grid-point method (ZSP-NGP) by Birdsall and Fuss [162].

Vorozhtsov [227] studied the use of sub-cell-sized spherical and square particles to ease the transition of a particle from one cell to its neighbor. Despite an improvement in the smoothness of solutions to the 1-D Riemann problem, the method still failed to account for any particle motion completely within each cell.

Cloud-In-Cell

Bi- (tri-) linear interpolation, also called Cloud-in-Cell (CIC) M_2, area-weighting, or square-particle, is the tensor product of the one-dimensional tent function which itself is the convolution of two top-hat functions. It is second order.

$$W(x) = \begin{cases} 0 & : \quad |x| > \varepsilon \\ 1 - |x| & : \quad |x| \leq \varepsilon \end{cases} \tag{2.9}$$

It was created by Birdsall and Fuss [162] in 1969 for use in plasma simulations. It was the method used by Christiansen [166, 167] in the first VIC method. Both Christiansen and Baker [201] suggest that use of these anisotropic interpolation kernels leads to fine-scale error, but that the error may not seriously

affect the large-scale features. A CIC interpolation method conserved total circulation and linear impulse, but not angular impulse, which has an error bound given in [201]. Brecht and Ferrante [203] use this area-weighting function with a digital filter and mid-range boost to reduce noise.

Λ_2

A third-order interpolation formula created not from the family of smooth formulae is the third-order Λ_2 [228].

$$\Lambda_2(x) = \begin{cases} 1 - x^2 & : \quad 0 \leq x < \frac{1}{2} \\ (1-x)(2-x)/2 & : \quad \frac{1}{2} \leq x < \frac{3}{2} \\ 0 & : \quad |x| \geq \frac{3}{2} \end{cases} \tag{2.10}$$

This filter function is not smooth or even continuous, but still maintains the first three invariants. Najm *et al* find that using this filter for redistribution (re-gridding) introduces spurious noise into their Eulerian calculation of the viscous term. Those authors use the smoother W_4 kernel for redistribution. Similarly, Cottet *et al* [46] find the same, and propose M4' or Λ_3, both 4-th order kernels.

M3

Also called the triangular-shaped cloud (TSC), this is a second order interpolation technique that arises from successive convolutions of the top-hat filter (just like CIC 2.4.2 and M4' 2.4.2). The formula for TSC is given in [229] as

$$M_3(x) = \begin{cases} 1 - c - x^2 & : \quad 0 \leq x < \frac{1}{2} \\ \frac{1}{2}(c - |x| + x^2) & : \quad \frac{1}{2} \leq x < \frac{3}{2} \\ 0 & : \quad |x| > \frac{3}{2} \end{cases} \tag{2.11}$$

Since its inception, it has been improved via "charge sharing," "subtracted dipole scheme" [230] which sets $c = 0$, and a scheme whereby the value of c is chosen to minimize noise in the one-dimensional spectrum [229] ($c = 17/60$).

Λ_3

This is a piecewise cubic fourth order function that is continuous, and was used by Cottet *et al* [46] for regridding. It seemed to behave as well as the

similar-stenciled M4' kernel in that regard.

$$\Lambda_3(x) = \begin{cases} (1-x^2)(2-|x|)/2 & : \quad 0 \le |x| < 1 \\ (1-|x|)(2-|x|)(3-|x|)/6 & : \quad 1 \le |x| < 2 \\ 0 & : \quad |x| \ge 2 \end{cases} \tag{2.12}$$

M4'

The M_4' (W4) [231, 232] method is used in smoothed particle hydrodynamics as well as by other vortex methods researchers.

$$M_4'(x) = \begin{cases} 0 & : \quad |x| > 2\varepsilon \\ \frac{1}{2\varepsilon}(2-|x|)^2(1-|x|) & : \quad \varepsilon \le |x| \le 2\varepsilon \\ 1 - \frac{5x^2}{2} + \frac{3|x|^3}{2} & : \quad |x| \le \varepsilon \end{cases} \tag{2.13}$$

The method is not strictly positive, as are the other methods, though it is C^1 smooth. Evaluation of this kernel requires knowledge of four grid points in each dimension, making a 64 point stencil in 3D. For that effort, M_4' rewards the user with fourth order accuracy.

Gaussian core

Alternatively, a Gaussian form of the core function can be made. Leonard [69] proposed a third-order Gaussian core function that was later shown to yield a second-order discretization by Beale and Majda [225]. Those authors continue to develop kernels of 4th and 6th order for 2D and 3D by combining scalings from lower-order exponential or Gaussian kernels. Hald [185] introduces several infinite-order cutoff functions. None of these functions provide compact support, a necessary requirement for fast methods, such as FMM [181] or MLC [173]. Ghoniem *et al* [81] used this second-order Gaussian core

$$f(r) = \frac{1}{\pi}\, e^{-r^2} f(r,R) = \frac{1}{\pi^{3/2}R}\, e^{-\frac{r}{R}^2} \tag{2.14}$$

on the vorticity in a 2D particle method. Nordmark [42] presents an 8th order cutoff function with compact support. Marshall and Grant [38] use the second form above, but state that any of a wide variety of spherically-symmetric, doubly- differentiable function would be acceptable.

Peskin function

Peskin [233] proposed a sinusoidal-based function that was used later for VIC [96, 149] and level-set [122] (with $\varepsilon = 4\Delta x$) formulations. Tryggvason [96] still noticed grid effects in the solution, which could be due to the rectangular nature of the three-dimensional application, or to the inherent problems in grid-based methods. Unverdi and Tryggvason [234] used $\varepsilon = 2\Delta x$. The present work uses the rectangular 3D Peskin kernel with $\varepsilon = 3\Delta x$.

$$\delta_\varepsilon(x) = \begin{cases} \frac{1}{2\varepsilon}\left[1 + cos\left(\frac{\pi x}{\varepsilon}\right)\right] & : \quad |x| \leq \varepsilon \\ 0 & : \quad |x| > \varepsilon \end{cases} \tag{2.15}$$

Other kernels

Wee and Ghoniem [235] present a remeshing filter that accounts for viscous dissipation.

Filter design

A proper filter must be not only conservative, but should be compact. Strict positiveness limits a filter to second order. Allowing a kernel to go negative lets you create filters of arbitrary order. The M4' method [231] is third-order accurate.

It was shown by Fureby [236] that only filtering operations with rotational symmetry will preserve material frame indifference, which is a desirable property of a subgrid model. A filtering of the vorticity field in wavespace can provide this spherical symmetry (shown by S. S. Wang, Ph. D. thesis, Stanford University Institute for Plasma Research Report 710, 1977).

Couët [22] claims that the grid dissipation in a VIC method is equivalent to LES-like subgrid-scale dissipation and uses a cutoff filter in Fourier space to smooth the aliasing created by rectangular interpolation kernels. The problem with calculating in Fourier space is that only periodic boundary conditions can be used—no internal boundaries can be present unless the method is modified. The advantage is that the filtering of high-wavenumber details is equivalent to subgrid-scale dissipation, or regularization (really).

Beale and Majda [225] design core functions with arbitrarily-high spatial order.

Fureby [236] states that a true SGS model must be frame- and Galilean-invariant, and thus require a filter kernel with rotational symmetry, but that

discretizing the LES equations onto a fixed grid breaks the rotational symmetry and reduces the reliance on a spherical filter. Are these same arguments valid for a vortex method?

Several simple rules can be used in filter construction, they are presented frequently in the literature [46, 184]

Filter comparison

Tryggvason [149] compares Christiansen's [166] 4-point filter to a more isotropic 2D filter based on the work of Peskin [233], which consists of a product of two 1-D cosine functions.

Winckelmans introduces a high order algebraic smoothing [36] that is of order equal to Gaussian smoothing kernels.

Ebiana [237] discusses numerical filters for VIC methods, but only studies 2D interpolation, and studies no spherical filters.

Mansfield, *et al* [238], in his study of particle filters and their usefulness for LES, defines a standardized filter size, and does two-level filtering for LES with particle filters instead of grid filters.

2.4.3 Domain decomposition

All fast velocity evaluation methods group individual elements together to facilitate faster computation. In addition, many computational methods related to Lagrangian method require quick searches of nearby elements, such as diffusion, or remeshing. In this section, we will discuss the various methods used to subdivide the spatial domain in Lagrangian vortex methods.

Regular subdivisions

VIC methods have traditionally grouped elements (more correctly their properties) onto a rectangular fixed grid. This facilitates solution of the Poisson equation using a method based on FFTs. There is no reason that VIC methods cannot use multigrid solvers for this solution, in which case an octree domain decomposition method may serve better.

Some authors fit a conformal grid to the surface of their model and solve the Poisson equation (or Navier-Stokes, or diffusion equations) on it.

Hierarchical subdivisions

FMM and treecode algorithms make extensive use of hierarchical binary or octree domain decomposition methods, especially insofar as the methods are commonly created for parallel computers.

Binary tree A binary tree representation subdivides any group along a specific (and usually axis-aligned) plane into two equal-sized groups. A box representing the initial (parent) group is saved, along with boxes for—and links to—the two new (child) groups. At the finest level, each box/group should contain the same number of elements (usually 30-500).

Quad-tree or oct-tree (B-tree, or N-ary tree) An octree differs from the binary tree in the respect that a parent box is subdivided into 8 child boxes, usually by geometrically bisecting a cubical volume that contains all of the box's elements. As a result, some of the 8 child boxes may have no elements at all, and the finest level of boxes will not have an equivalent number of elements. In fact, the depth of the tree will vary throughout the domain, depending on element density. An advantage of this method is that neighboring boxes can be determined algorithmically, sometimes very quickly by using a hash table (Warren and Salmon?).

Box shrinking In both cases, a group's box can be shrunk down to the minimum bounds necessary to contain the group's elements. This offers a significant advantage, as the box opening criterion in many FMM and treecodes depends on the ratio of the box's size to the distance from the box center to the target. Greater accuracy or faster run times result. This method was introduced by Clarke and Tutty [218].

Clarke and Tutty [218] presents a binary tree that shrinks boxes at each level, before splitting to create the next level. The same method is used again in 2D [239] and extended to 3D [190, 82].

2.4.4 Vorticity divergence

Move this to "particle" section.

Are there two kinds of vorticity divergence? (initial discretization and change due to stretch)

Generally, a particle representation of a vorticity field is not necessarily divergence-free [240]. "Relaxation" methods have been devised to counter this behavior, and can take one of a number of forms [241, 36, 44]. The relaxation method in [36] does not conserve energy, helicity, or angular impulse until a PSE scheme for vorticity diffusion is added. Marshall and Grant [38] solve a global matrix equation to reset the particle strengths to recreate a divergence-free vorticity field.

Marshall and Grant [38] comment on vorticity divergence (and it seems that they do not separate the two source of divergence, as listed above). Any field can be made divergence-free by subtracting the gradient of an unknown scalar field. For vorticity, this unknown scalar field may be non-zero, but still does not enter into the velocity calculations! It would, however, affect the vorticity evolution equation. Despite that, the authors compare simulations identical in every respect save for the addition or omission of their divergence-free vorticity step and determine that the results are nearly identical. The numerical algorithm with divergent vorticity (and the one most commonly used) can be shown to conserve vorticity divergence to first order in Δt.

Vorticity divergence is contained in recent calculations, but is nevertheless present [37]. Nontheless, mathematically, both VIC formulations (vorticity-streamfunction and vorticity-velocity) return divergence-free velocity fields from non-divergence-free vorticity fields.

Chatelain and Leonard [55] mention divergence error in their particle-based redistribution scheme. Their errors are on the order of $1.0e-2$.

2.4.5 Computing derivatives

How does one do this in a Lagrangian calculation? The PSE scheme must calculate derivatives, Eulerian schemes can use finite differences on a grid. Eldredge [242] presents a method for deterministic treatment of derivatives in particle methods.

An additional complexity involves calculating derivatives of a quantity on a surface, especially a triangulated surface. Think of the surface tension effect.

Just as the VIC method uses a temporary grid to determine the flow velocity, some particle vortex methods use a temporary, superimposed grid to perform some calculations. Marshall and Grant [38] and Liu [39] use a grid to compute stretch. Others have used grids to compute vorticity diffusion [40].

2.4.6 Generalized Helmholtz decomposition

The generalized Helmholtz decomposition (GHD) links the boundary conditions and interior volume of a flow. It contains integrations over the vorticity-containing volume of the flow and over the vortex-sheet-strength-containing boundaries of the flow.

$$\eta(\boldsymbol{x})(\boldsymbol{u}(\boldsymbol{x}) - \gamma(\boldsymbol{x}) \times \boldsymbol{n}) = \frac{\boldsymbol{\omega}(\boldsymbol{y}) \times \boldsymbol{r}}{r^2}\, d\Omega(\boldsymbol{y}) + \frac{\boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{y})}{r^2}\, d\Gamma(\boldsymbol{y}) \tag{2.16}$$

It is used in Ingber [243] to solve 2D thermal flows inside enclosures. References can be found therein.

2.4.7 Diagnostics/Conserved quantities

There are a number of flow quantities whose value should remain constant throughout the flow's evolution. These are often useful indicators of the accuracy of a simulation and their conservation is equated with validation of a numerical scheme.

The invariants of an inviscid flow with no energy input are the circulation, linear impulse, angular impulse, kinetic energy, and helicity. Additionally, the volume enclosed by the tracked front should be constant. Many of these flow invariants can be easily calculated via summations over the vorticity field according to the following relations [65, 35]:

$$\Gamma_i = \int_{\mathcal{V}} \omega_i \, d\boldsymbol{x} \tag{2.17}$$

$$\boldsymbol{I} = \frac{1}{2} \int_{\mathcal{V}} \boldsymbol{x} \times \boldsymbol{\omega} \, d\boldsymbol{x} \tag{2.18}$$

$$\boldsymbol{A} = \frac{1}{3} \int_{\mathcal{V}} \boldsymbol{x} \times \boldsymbol{x} \times \boldsymbol{\omega} \, d\boldsymbol{x} \tag{2.19}$$

$$H = \int_{\mathcal{V}} \boldsymbol{u} \cdot \boldsymbol{\omega} \, d\boldsymbol{x} \tag{2.20}$$

Alternate expressions appear in [36].

Enstrophy, $\varepsilon = \frac{1}{2} \int_\Omega \boldsymbol{\omega} \cdot \boldsymbol{\omega} \, \mathrm{d}\Omega$, is not conserved, but is related to kinetic energy, $E = \frac{1}{2} \int_\Omega \boldsymbol{u} \cdot \boldsymbol{u} \, \mathrm{d}\Omega$, is the following way [192].

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -2\nu\varepsilon \tag{2.21}$$

Kudela and Regucki [178] show worst-case final energy error and total helicity of 40% and 10^{-2} for a 3D inviscid method with very small step sizes. Cottet [184] also reports errors, but with a 2D viscous method. Complete presentations of these and other errors appear in Brady *et al* [87] and Stock [244]. These numbers are infrequently reported.

A measure of numerical diffusion emerges from the relationship between energy and enstrophy, and is given in [49] as

$$\nu_{effective} = \frac{1}{2\varepsilon}\frac{\mathrm{d}E}{\mathrm{d}t} \tag{2.22}$$

2.4.8 Forces on bodies

Several methods exist for computing forces on bodies in vortex flows. The canonical method considers derivatives of the linear impulse:

$$\frac{\boldsymbol{F}}{\rho} = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int_V \boldsymbol{x}\times\boldsymbol{\omega}\,dV\right] = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\sum_{i=1}^{N}\boldsymbol{x}_i\times\boldsymbol{\alpha}_i\right], \tag{2.23}$$

A method that will return the forces and moments on individual bodies from a collection of bodies relies on being able to define control volumes in the fluid domain around each body. The lengthy equation, given in [37] from an earlier paper [245], requires derivatives of the velocity gradient. A summary of methods appears in [246].

Chapter 3

The Vorticity Equation

The curl of the compressible Navier-Stokes equation (and a more complete version of the vorticity evolution equation) is called the Helmholtz equation, and is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \underbrace{-(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{\omega}}_{advection}\ \underbrace{+(\boldsymbol{\omega}\cdot\nabla)\,\boldsymbol{u}}_{vortexstretching}\ \underbrace{-\boldsymbol{\omega}\,(\nabla\cdot\boldsymbol{u})}_{dilatation}\underbrace{+\frac{1}{\rho^2}(\nabla\rho\times\nabla p)}_{baroclinic}$$
$$+\frac{\mu}{\rho^2}(\nabla\rho\times\nabla\times\boldsymbol{\omega}) - \frac{4\mu}{3\rho^2}\big[\nabla\rho\times\nabla(\nabla\cdot\boldsymbol{u})\big]\quad\underbrace{+\frac{\mu}{\rho}\nabla^2\boldsymbol{\omega}}_{viscousdiffusion}\quad+$$
$$\left\{\nabla\times\left[\frac{1}{\rho}\Big(-\frac{2}{3}(\nabla\cdot\boldsymbol{u})(\nabla\mu)+2(\nabla\boldsymbol{u})\cdot(\nabla\mu)+(\nabla\mu)\times\boldsymbol{\omega}\Big)\right]\right\} \tag{3.1}$$

which is an expansion of this simpler form

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \underbrace{-(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{\omega}}_{advection}\ \underbrace{+(\boldsymbol{\omega}\cdot\nabla)\,\boldsymbol{u}}_{vortexstretching}\ \underbrace{-\boldsymbol{\omega}\,(\nabla\cdot\boldsymbol{u})}_{dilatation}\underbrace{+\nabla p\times\nabla\frac{1}{\rho}}_{baroclinic}$$
$$\underbrace{+\nabla\times\boldsymbol{f}_e}_{externalforces}\ \underbrace{+\nabla\times\left(\frac{1}{\rho}\nabla\cdot\bar{\bar{\tau}}\right)}_{viscousdiffusion}. \tag{3.2}$$

Introducing the material derivative, and simplifying the viscous diffusion term for the case of a Newtonian, incompressible fluid with constant kine-

matic viscosity (ν), we have the common vorticity equations

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\,\boldsymbol{u} + \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nabla\times\boldsymbol{f}_e + \nu\nabla^2\boldsymbol{\omega}, \quad (3.3)$$

which in the absence of viscosity (Euler limit) and other external forces become

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \underbrace{-(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{\omega}}_{advection}\ \underbrace{+(\boldsymbol{\omega}\cdot\nabla)\,\boldsymbol{u}}_{vortexstretching}\ \underbrace{-\boldsymbol{\omega}\,(\nabla\cdot\boldsymbol{u})}_{dilatation}\ \underbrace{+\frac{1}{\rho^2}(\nabla\rho\times\nabla p)}_{baroclinic} \quad (3.4)$$

A formulation more amenable to vortex sheet modeling is the evolution equation for vortex sheet strength (eqn. 3.16 in [89]).

$$\frac{D\boldsymbol{\gamma}}{Dt} = \underbrace{\boldsymbol{\gamma}\cdot\nabla\boldsymbol{u}^{PV}}_{stretch}\ \underbrace{-\boldsymbol{\gamma}(\boldsymbol{P}\cdot\nabla\cdot\boldsymbol{u}^{PV})}_{dilatation}\ \underbrace{+2\,A\,\boldsymbol{n}\times(\bar{\boldsymbol{a}}-\boldsymbol{g})}_{baroclinic}\ \underbrace{+\frac{4T}{\rho_1+\rho_2}\boldsymbol{n}\times\nabla\kappa_m}_{surfacetension} \quad (3.5)$$

The identical formulation for the circulation is as follows (eqn. 3.15 in [89]).

$$\frac{D\Gamma}{Dt} = \underbrace{2\,A\left(\frac{D\phi^{PV}}{Dt} - \frac{1}{2}|\boldsymbol{u}^{PV}|^2 + \frac{1}{8}|\Delta\boldsymbol{u}|^2 - \boldsymbol{g}\cdot\boldsymbol{x}\right)}_{baroclinic}\ \underbrace{+\frac{2T}{\rho_1+\rho_2}2\kappa_m}_{surfacetension} \quad (3.6)$$

3.1 Vortex stretching

In the basic three-dimensional vortex particle method, the vortex stretching term is accounted for by calculating $\nabla\boldsymbol{u}$ at the location of the particle. This has been shown to create vorticity fields that are not solenoidal. Alternative methods have been proposed to rectify this problem. Cottet [184] provides a summary of these methods.

Knio and Ghoniem [72] compare the transpose scheme to the non-transpose scheme and find that the non-transpose scheme works better, but that was caused by the relatively low number of large-cored elements in the simulation. Other simulations done using higher-order smoothing functions [33] have found the transpose scheme performing better. Mixed schemes combining both have been proposed [72, 184].

Knio and Ghoniem [86] use two different methods to compute the stretch term. In simulations without a baroclinic source term in the vorticity equation, the stretching term is computed based on the change in length of the sides of a rectangular element that are oriented parallel to the vorticity vector. When a density interface exists, the $\nabla \boldsymbol{u}$ term is found by analytically differentiating the desingularized Biot-Savart law.

Alternatively, vortex filament and vortex sheet methods automatically includes the effects of vortex stretching and generates solenoidal vorticity fields.

3.1.1 Particle discretization

The interaction between local vorticity vectors and the velocity gradient causes vorticity stretch, a term in the three-dimensional vorticity evolution equation. While filament and sheet methods intrinsically satisfy this term, vortex particle methods must apply it to each particle. This calculation can use a finite-difference approximation (Beale and Majda proof [29] and method [32]), or can be calculated by differentiating the regularized velocity kernel, as was first proposed by Anderson and Greengard [247], proved by Beale (transpose method) [33], and demonstrated by Fishelov [57]. VIC methods have an intrinsic advantage in that the vortex stretching can be calculated directly on the temporary Eulerian grid, as shown by many authors, lately Liu and Doorly [39], but also [38].

Vortex filament [12, 69, 71] and vortex sticks (early papers by Chorin [28]) use an element-wise approach to find the velocity gradient along the vorticity vector. For singular particles, it is shown that the transpose scheme works best numerically [36].

Brecht and Ferrante [202] track connectivity between particles in order to maintain volume/area and, thus, implicitly update the particles' strengths to account for vortex stretching. Any scheme that maintains connectivity can do this on an element-by-element basis. Is this for real? In [203] it appears that they do nothing of the sort.

3.1.2 Filament discretization

The change in a filament's total vorticity due to vortex stretching is automatically accounted for if the circulation is held constant and the filament element is connected to its neighboring two elements via two common nodes.

3.1.3 Sheet discretization

The change in a triangular element's total vorticity due to vortex stretching is automatically accounted for if the circulations on the element edges are held constant and the triangular element is connected to its neighboring three elements via three common nodes and edges.

3.2 Diffusion methods

Dissipation is an intrinsically different procedure than regularization, though tests [161] have shown that the two behave very similarly in certain cases. This is because regularization in a vortex method limits the magnitude of the velocity gradient, which in turn prevents the stretching term from creating the required new vorticity. This section will describe diffusion methods, as regularization and its effects are covered in section 2.2.1.

Chorin describes the "artificial viscosity" imposed by regularization in [19], and in that work it was imposed to eliminate the singularity in the vortex sheet description.

The need for some diffusion model is greatly enhanced by the additional physics of three-dimensional flows. Vortex stretching can produce extremely high, intermittent, local vorticity magnitudes. If a model does not regularize or dissipate these large vorticity gradients, it will produce unphysical results, or fail to continue altogether.

A viscous diffusion or subgrid dissipation model can take the form of either a) damping of the geometry of the smallest details in the resolved flow, or b) damping, reducing, or exchanging the strength of elements.

Two-dimensional flows can survive aggressive vorticity manipulation and return similar dynamics. 3D flows have no similar flexibility.

Good discussions of competing methods are found in Shankar [248] and Gharakhani [93].

A righteously awful method [249] consists of limiting the maximum vorticity for each vortex particle based on the expected maximum flow vorticity for the given Reynolds number.

Lozano *et al* [88] included an artificial viscosity term in the vortex strength evolution equations simply to smooth the unstable advective terms.

Most schemes for viscosity in vortex methods use a technique called "operator splitting" [250], which numerically splits the vorticity update equation

into separate convection and diffusion steps.

3.2.1 Random walk

The Random Vortex Method (RVM) was introduced by Chorin [250] to study slightly viscous flows. The RVM uses a Wiener process to perturb the motion of each vortex particle, which simulates diffusion of vorticity. It is easy to implement in flows with solid boundaries. Leonard [12] references work that shows that to achieve accurate results of viscous diffusion, the RVM requires a large number of particles compared to the Reynolds number. The method was proven to converge to the heat equation by Hald [251] and by Puckett [252].

A random-walk method is used by Gharakhani and Ghoniem [204, 93]. Savoie, Gagnon, and Mercadier [171] also used the random walk method, but to compute the starting flow behind a two-dimensional step. A numerical study of the convergence of this method in two dimensions is presented by Mortazavi [253]. That paper contains references to a number of other studies of RVM's convergence. A two-dimensional random vortex method is used by Gagnon [254] to simulate the flow over a single back-facing step and a double symmetrical backward-facing step. Turbulence statistics and dominant frequencies are compared to real flows. Abdolhosseini [172] studied the turbulence statistics in a uniformly sheared flow with a two-dimensional VIC method using RVM. The results weren't good. The same technique was used for the spatially-growing mixing layer [255]. Other uses appear in [143].

Marshall and Grant [256] construct a diffusion velocity method for axisymmetric flows. Milane [257] uses a diffusion velocity method to compute LES solutions in a 2D mixing layer.

The problem with RVM is the same as with VIC: there is regularization, but the elements do not change their strength. That is the inherent problem that is solved by PSE, VRM, and other newer methods.

Additionally, the RVM suffers from low-order non-uniform convergence due to its stochastic character.

Discussions with Gharakhani reveal that RVM can introduce unphysical small-scale oscillations in the solution.

3.2.2 Core-spreading techniques

Kuwahara and Takami [20], in an early point-vortex method, used vortex blobs with spatially-varying cores, the goal of the latter was to simulate viscous diffusion. Leonard [12] later used the method.

The core-spreading technique derives from the solution to the two-dimensional Navier-Stokes equations in vorticity coordinates for the initial conditions

$$\omega(r, 0) = \Gamma_0 \, \delta(x) \, \delta(y). \tag{3.7}$$

In this case, the Navier-Stokes equations are simply

$$\frac{\partial \omega}{\partial t} = \nu \, \nabla^2 \omega. \tag{3.8}$$

The exact solution for the evolution of the total circulation is

$$\Gamma = \Gamma_0 \, (1 - e^{-r^2/4\nu t}) \tag{3.9}$$

Thus, one can approximate the decay of a 2D vortex particle or a 3D vortex filament by reducing the effective circulation or increasing the effective radius of the filament/particle using the solution above.

While this method contains numerical inconsistencies (Peter Bernard's wording), or that core-spreading approximates the wrong equation [258], in the limit of an infinite number of particles (Moeleker's wording), those problems can be corrected by instantaneous reconfiguration of large vortex blobs to thinner ones [259].

Advantages of the core-spreading technique are summarized in Rossi [259], primary is their fully-deterministic character, lack of reliance on operator splitting, and freedom from flow geometry considerations.

A vortex sheet analog of core-spreading is demonstrated in Tryggvason *et al* [159], which used a quasi-1D method developed for mixing and combustion [260, 261] called LIM. This is also described in section III of Beigie, Leonard, and Wiggins [262].

3.2.3 Hairpin removal

Hairpin removal is a renormalization process by which small-scale detail is removed via a local mesh redistribution algorithm. It was introduced by Feynman [263] in 1957, and elaborated upon by Leonard [68] and Chorin

[264, 265], eventually being extended [266] with a renormalized Biot-Savart interaction to account for the hairpin removal. Leonard [68] suggests that the bookkeeping will be troublesome.

Fernandez [187] utilizes a similar method, called "filament surgery," with a parallel treecode to study the vortex collapse-reconnection process. There is also a good description of the early development of the method of remeshing via merging.

This viscous-like scheme is most closely related to the current method of merging in *vort3d*.

3.2.4 Particle Strength Exchange (PSE)

Also called "circulation redistribution" methods, these are deterministic, as opposed to stochastic, procedures. These were introduced by several authors, see [248] for details. Mas-Gallic [267] first proposed this method in 1987.

Degond and Mas-Gallic investigate diffusion operators for particle methods in rigorous detail for isotropic [268] and anisotropic [269] viscosity. This is the origin of PSE, and these authors are commonly credited with its creation. In this scheme, the diffusion operator (the Laplacian) is approximated by an integral operator, which is in turn discretized over particles in the local area.

A related deterministic method was introduced by Fishelov [270] which also replaced the diffusion operator by an integral one, but due to its construction can support higher-order cutoff functions.

The Particle Strength Exchange (PSE) model [36] is a method to account for viscous diffusion in a particle vortex method. It is a correction term to each particles' vorticity, and is usually calculated as a separate step, via operator splitting. It has been shown to be a weak solution to Navier-Stokes.

$$\frac{\partial \boldsymbol{\omega}_p^h}{\partial t} = \frac{\nu}{\varepsilon^2} \sum_q \left(v_q \boldsymbol{\omega}_q^h - v_q \boldsymbol{\omega}_p^h \right) \eta_\varepsilon (\boldsymbol{x}_q^h - \boldsymbol{x}_p^h) \tag{3.10}$$

This equation governs the exchange of vorton strength $v\,\boldsymbol{\omega}^h$, the product of vorton volume and vorticity, between neighbors based on a viscosity ν, cutoff function η and radius ε.

Variable-core-sized vortex blobs can be accommodated in the PSE scheme by remapping the variable blobs onto uniform blobs. This is first mentioned in [46], and appears in [184].

Cottet [10] remeshes *before* the diffusion step in order to avoid quadrature errors.

Eldredge [242] extends the PSE scheme to create a general method for finding derivatives in particle methods, including one-sided derivatives for use in hyperbolic problems.

Other authors have used the PSE scheme in their vortex particle computations [192].

3.2.5 Vorticity Redistribution Method (VRM)

VRM [248] is different from PSE in that it can conserve moments to arbitrarily-high orders of accuracy, can fill holes caused by excess strain, and does not require remeshing of the particles on to uniform grids to maintain accuracy. It does this by solving an under-determined system of equations for the N closest particles within a radius of $C\sqrt{\nu t}$. The system is assembled by requiring local conservation of circulation and momenta. If a solution is not achieved, particles are added, and the problem is restated and a solution is attempted again.

Shankar and van Dommelen [248] propose a related vorticity redistribution method that limits the maximum distance that the circulation of a vortex can move. It requires six local vortices for first order accuracy, and more for higher orders. Unlike the schemes of Fishelov [270] and Mas-Gallic [268], this scheme attains positivity (no false reversed vorticity is ever created). Their method is slow, though—on the order of the convection calculation.

Both VRM and PSE rely on effective search strategies to identify all particles within a given distance (say, r) from a point. Proper spatial subdivision and use of search trees can cut this down to $\mathcal{O}(logN)$ time per search, where one search must be done for each particle. One possible method to speed this up would be to perform a large search for all particles within a $r + \Delta$ radius, and use that list for the VRM calculations for all particles within a Δ radius. Thus, the search is differentially larger, but the results of the search can be reused for as many particles exist in the Δ-radius sphere. Other search strategies involve organizing the data within each branch of the tree along its principal axis [271].

VRM is used in 2D flow over a cylinder [239]. Lakkis and Ghoniem [272] use VRM with variable core sizes for combustion simulations of a vortex ring. A higher-order VRM is described in Gharakhani [273]. VRM is used in the context of LES by Gharakhani [274].

3.2.6 Free-Lagrangian method

Detailed in [275], this method calculates the diffusion of a number of vorticies by constructing a Voronoi diagram and calculating finite differences between neighboring regions. The method is only weakly first order and requires uniformity among the particle positions.

See also [276, 277, 108].

3.2.7 Eulerian formulations

Instead of complicated particle operators, one could solve for the diffusion term in the velocity or vorticity equation explicitly on an overlaid Eulerian grid ($\nabla^2\omega$). This method may have been first used within a 2D VIC method by Graham [278] An easier reference to acquire is Najm [40], 1993. This method lends itself to VIC methods, and other methods where the particles are reassigned periodically to a regular grid.

Cheng [177] uses a "diffusion-vortex" scheme, proposed by Lu and Ross [279] in 1991, within a 2D VIC solver to solve for flow over a circular cylinder.

VIC methods have an intrinsic advantage in that many terms in the vorticity equation can be calculated directly on the temporary Eulerian grid, as shown by many authors, lately Liu and Doorly [39].

Ould-Salihi [11] suggests that while 2nd order interpolation operators are adequate for pure VIC methods, they create unacceptable numerical diffusion when used to explicitly solve for diffusion on a grid, or to be used in vorticity boundary conditions. A 3rd order interpolant is used for the vorticity because its Laplacian is later calculated by finite differences. Even though this method is not conservative, the regularity imposed by the regridding step serves to make the lack of conservation not noticeable. Najm *et al* [43] found very much the same thing, but earlier, in their FMM calculations for reacting flow modeling. They found that the third-order (but C_0 continuous) Λ_2 interpolation kernel introduced spurious noise in the viscous source terms (which used 3rd order derivatives), and were forced to upgrade to the smoother W_4 kernel [232].

Akbari and Price [200] use a second-order ADI scheme for the diffusion term and a 2D VIC method for the convection term in their simulations of an oscillating wing section.

3.2.8 Other methods

Cottet [9] uses a remapped grid, in which certain areas have higher (near boundaries) or lower (far-field wake) grid density, for particle remeshing. They claim that that this is equivalent to Ghosal and Moin's [280] spatially-varying LES filter sizes.

3.3 Large Eddy Simulation (LES)

Just as the Navier-Stokes equations in velocity form (1.1-1.3) can be split into grid-resolved and subgrid scales, so, too, can the vorticity equation. With an overbar representing a filtered quantity, the filtered vorticity equation is

$$\frac{D\,\overline{\omega}_i}{Dt} = \overline{\boldsymbol{\omega}} \cdot \nabla \overline{u}_i + \nu \nabla^2 \overline{\omega}_i + \frac{\partial}{\partial x_j}(\Phi_{ij} - \Phi_{ji}), \tag{3.11}$$

with

$$\Phi_{ij} = \overline{\boldsymbol{\omega}_i \boldsymbol{u}_j} - \overline{\boldsymbol{\omega}}_i \overline{\boldsymbol{u}}_j \tag{3.12}$$

representing the Helmholtz, or vorticity, stresses.

Some vortex methods, by their construction, exhibit subfilter-scale dissipation. Most are able to account for it explicitly.

Chorin [266] discusses the microstructure of vortex filaments. Moeleker and Leonard [54] introduce a tensor-diffusivity subgrid-scale model for the incompressible scalar advection-diffusion equation, but it seems like all of the problems that prompted that method are addressed by LIM. Couët [22] claims that the grid dissipation in a VIC method is equivalent to LES-like subgrid-scale dissipation and uses a cutoff filter in Fourier space to smooth the aliasing created by rectangular interpolation kernels.

Mansfield *et al*[48], proposes a Smagorinsky-equivalent subgrid-scale model for the filtered vorticity equations. It shows that if the Helmholtz stresses are represented as

$$\Phi_{ij} = \nu_T \frac{\partial \overline{\boldsymbol{\omega}}_i}{\partial x_j} \tag{3.13}$$

where the eddy diffusivity ν_T is

$$\nu_T = (c_T \delta)^2 \sqrt{2 \overline{S}_{ij} \overline{S}_{ij}} \tag{3.14}$$

and the constant in the eddy diffusivity equation can be taken as $c_T = 0.15$, similar in magnitude to the Smagorinsky model in velocity variables, then the evolution equation for vorticity is

$$\frac{D\,\overline{\omega}_i}{Dt} = \overline{\boldsymbol{\omega}} \cdot \nabla \overline{u}_i + \nu \nabla^2 \overline{\omega}_i + \nu_T \left(\frac{\partial \overline{\omega}_i}{\partial x_j \partial x_j} - \frac{\partial \overline{\omega}_j}{\partial x_i \partial x_j} \right). \tag{3.15}$$

The authors also mention the following.

> Note that Φ_{ij} represents subgrid-scale vortex stretching and tilting due to unresolved motion, while Φ_{ji} reflects vortex transport by subgrid-scale velocity fluctuations.

Mansfield *et al*[238], proposes a dynamic eddy-diffusivity LES model in the vorticity variables based on two-level filtering of the vorticity field and compares vortex methods to spectral methods in a simulation of homogenous isotropic turbulence. The scheme uses an eddy diffusivity model that, by construction, creates no subfilter torque.

More applicable is Cottet [281], which introduces a Lagrangian method for LES consisting of an anisotropic and less-diffusive method. The work leverages the truncation error normally produced by unremeshed vortex methods. Using the vorticity formulation of the Navier-Stokes equations, Cottet *et al* [282] describe two new LES formulations, one based on the vorticity angles. Vortex methods are not used?

Gharakhani [274] presents an application of the vorticity redistribution method [248] for large-eddy simulation.

Milane [257] uses a diffusion velocity method to compute LES solutions in a 2D mixing layer.

A large question remains: how does one accommodate the subgrid-scale modeling of a sheet of discontinuity? It is even possible? Lundgren *et al* [283] make an interesting observation:

> The roll-up of unstable Helmholtz vorticies [has] effectively produced a thicker interface... It is our point of view that by computing with larger [values for Krasny's] δ we simulate the effect of averaging over these 'turbulent' fluctuations.

3.4 Vorticity creation at walls

If viscosity is present in a flow, all solid boundaries are eligible sites for vorticity creation.

3.4.1 Euler limit

To support inviscid internal boundaries, one normally has to solve a Boundary Element Method solution, with the constraint of zero normal flow through any solid surface.

Note that the inviscid case is different that the limit of viscous flow as $re \to \infty$, or the *Euler limit*. In the Euler limit, no-slip is still imposed on a solid surface, and a boundary layer of thickness $\delta \to 0$ is formed on the surface. Wu [197] says that the jump across this thin vortex sheet is different than the jump across the solid surface.

These internal boundaries can be made inviscid. Even so, it is still possible to introduce vortex shedding off sharp edges via a thin layer of vorticity. This mimics viscous shedding at high Re. Boundaries in inviscid flows do not normally create vorticity in a flow. The flow slips perfectly over the boundary.

The method of images is an inviscid method for boundary treatment, but only works for flat or spherical boundaries. It is used for calculation of field velocities, not for determining the strength of the shed vorticity.

Scorpio [284] used FMM, but only to accelerate a BEM calculation, and not for a vortex method. He did do it for a free-surface problem, though.

An additional concern is the pseudo-viscous approximation of creating layers of vorticity where separation and shedding would take place (to satisfy the Kutta condition at trailing edges). Leonard [12] references three works that address this. In addition, extensive discussion appears in [196, 184].

Free vortex filaments are shed from propeller blades in Politis [83].

3.4.2 Viscous

The treatment of viscous boundaries in a vortex method is done either by defining, at the boundary, special sheets or tiles of vorticity which stay at the boundary and diffuse their vorticity to free elements, or by directly shedding their created vorticity on to the free elements (usually particles).

Lighthill [285] introduced the "vorticity source strength," or later called the "boundary vorticity flux," per unit time and area as the product of the kinematic viscosity the the normal derivative of the vorticity: $\nu\,(\partial\boldsymbol{\omega}/\partial\boldsymbol{n})$. Applying the tangential component of the Navier-Stokes equations to the rigid wall, one can see that this flux also equals $(1/\rho)(\partial p/\partial x)$.

Chorin [250] introduces a method for creating vorticity at boundaries and proposes a split scheme for handling boundaries and diffusion together in a vortex method. The choice of placement of the initial diffusion of vorticity away from a wall was also studied. Leonard may have been the first to use vortex sheets to represent boundary layers [286], these layers separate to form free vortex filaments in the wake. Chorin [142] was also the first to propose using vortex sheets to represent the boundary layer, this leverages the boundary layer approximation that tangential derivatives are of smaller order than normal derivatives. These vortex sheets are generated and a random component added to their motion to simulate viscosity [28]. Fishelov [57] combined that with a purely Lagrangian method for solving for velocity gradients in order to solve for the flow over a flat plate.

Fishelov [57] uses a vortex particle method to solve for the flow over a flat plate using the vorticity form of the Prandtl equations for a thin layer $0 \leq z \leq z_0$ (the tile model, using sheets), and of the Navier-Stokes equations outside of that (the random-vortex method).

A simple method of adding vorticity due to ground interference into an axially-symmetric flow is given in [283].

Alternatively, a viscous boundary will create vorticity at its surface and release it into the flow. Methods must be created to allow the creation of vorticity of proper strength and position at each time step. A regarded implementation for viscous boundary conditions is from Koumoutsakos *et al* [287]. A good summary of methods created to solve this problem appears in [93].

Bernard [91] adapted Fishelov's scheme for boundary layer flows to use a vortex sheet method. Bernard [92, 144] uses a combined sheet/filament method to discretize the vorticity in his wall-bounded flow simulations. Many authors use discretized bound vortex sheet elements in the computation of 2D and 3D boundary-layer flows to track vorticity diffusion into the fluid [91, 92, 93].

Winckelmans [44] proposes a method for this. An entire chapter in Cottet & Koumoutsakos [184] is dedicated to the treatment of viscous boundaries in vortex methods.

Szumbarski and Wald [288] found a way to eliminate non-physical pressure fields which imposes additional constraints for vorticity production at the boundary.

Buron and Pérault [289] present a vortex method that is based on a fixed grid of triangular prisms, it uses a traditional BEM to solve for doublet strength.

Flexible immersed boundaries in a two-dimensional fluid are addressed by Cortez [129] via impulse blobs.

Ploumhans and Winckelmans [53] also corrects the PSE in the vicinity of a boundary to avoid spurious vorticity flux during the convection/PSE step (2D).

Ploumhans [37] modified the PSE scheme for diffusion in the presence of solid boundaries (3D).

Ould-Salihi *et al* [11] describes a viscous splitting algorithm (see references) to solve for the viscosity boundary condition which involves two applications of the PSE scheme.

3.5 Baroclinic generation

A pure vortex sheet is a special class of interface that exhibits no surface tension or baroclinic source terms, and, thus, maintains a constant vortex sheet strength over its entire surface. That restriction is broken here, too, by the inclusion of these terms.

The portion of the vorticity evolution equation 3.1 relevant to baroclinic generation due to density variation or shock passage is

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{1}{\rho^2}(\nabla\rho \times \nabla p) - \boldsymbol{\omega}(\nabla \cdot \boldsymbol{u}), \tag{3.16}$$

or, without the pressure variable [290]:

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{1}{\rho}\left[\nabla\rho \times \Big(- \frac{D\boldsymbol{u}}{Dt} + \boldsymbol{g}\Big)\right] - \boldsymbol{\omega}(\nabla \cdot \boldsymbol{u}), \tag{3.17}$$

and for incompressible flow, assigning $A = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$, we have

$$\frac{D\boldsymbol{\gamma}}{Dt} = 2\,A\,\boldsymbol{n} \times \Big(\frac{D\boldsymbol{u}}{Dt} - \boldsymbol{g}\Big). \tag{3.18}$$

In this section, mixing will be ignored. All fronts shall represent pure, immiscible interfaces.

Extensive formulations for interfacial vorticity dynamics are given by Wu [197]. In particular, Lugt (ref 11,12 in Wu) noticed that the surface vorticity does not provide information about the rate of boundary vorticity flux. Wu states that for $Re \gg 1$, solid-wall boundary vorticity and vorticity fluxes of $\mathcal{O}\left(Re^{1/2}\right)$ and $\mathcal{O}\left(1\right)$ must appear, but that for interfacial fluid-fluid boundaries, these reduce to $\mathcal{O}\left(1\right)$ and $\mathcal{O}\left(Re^{-1/2}\right)$, respectively. This is why potential-flow theory for water waves works well.

Multifluid research is common outside the field of vortex methods.

The problem of density stratification solved by a vortex sheet is remarkably similar to that of the Saffman-Taylor instability in Hele-Shaw cell flow defined by a viscosity discontinuity (instead of the Atwood ratio-defined density discontinuity), with the exception being the absence of an integro-differential equation. In this case, the Atwood number is replaced by the mobility ratio, and standard VIC methods can produce results [291, 292]. This was first pointed out by de Jong [293] in 1960, and first simulated by Meng and Thomson [56] in 1978. Tryggvason and Aref [291, 294] used a VIC formulation to simulate the flow for arbitrarily-large values of the viscosity "Atwood ratio". Another presentation of these formulas is in Hou [295].

A particularly interesting aspect of computational sheet methods for flows with fluid discontinuities is the motion of the sheet markers. Zufiria [296], and initially Baker *et al* [297], points out that the marker motion can follow the upper fluid, the lower fluid, or any value in between. The choice of this parameter affects the form of the vorticity evolution equation. Pozrikidis [89] uses this parameter for the 3D equations.

The mathematics of vector calculus on a surface is summarized in Wu [197], (Appendix A).

3.5.1 Weak stratification

The kinematic equations for the vortex sheet strength first appear (where?). Weakly stratified flows are those in which the full equations of vorticity generation are modified by the Boussinesq approximation, that $g \to \infty$ and $A \to 0$, but Ag remains finite. This can also be described as assuming that the pressure can be treated to only first order ($\nabla p = \rho_0 \boldsymbol{g} + \mathcal{O}(\frac{\nabla \rho}{p_0})$ [203]. First introduced in VIC formulations by Meng and Thomson [56] for the simulation of a buoyant cylinder and gravity current, and also used by Meng

[298] in a strange fashion. The same formulation was used for the Taylor-Saffman instability of flow through a porous medium (a problem frequently appearing in the literature [293]). Anderson [299] uses a direct method with exponential core functions to compute a 2D thermal.

Simulations of a vortex ring impinging on a density interface appear in Dahm, Scheil and Tryggvason [300], and even thought the Atwood number is small, there are some significant dynamical differences in the results when the circulation density changes. This is due to the tests having different Froude numbers (ratio of hydrodynamic to hydrostatic pressure gradients).

3.5.2 Strong stratification

Solution of a free-surface problem (the extreme example of strong stratification, but including essential features of all strong stratification methods) in the Lagrangian sense requires simultaneously solving for the position of the free surface and the dipole (vortex) or source strengths. The choice of the surface element type and the boundary condition required (Dirichlet or Neumann) determines whether the problem to be solved is a Fredholm integral equation of the first or second kind. See Wu [197] for a discussion of these types of boundary conditions for wall and free surface flows. See Baker *et al* [297] for the classification of Fredholm equations. The boundary integral equation that uses vortex sheet strength as the unknown and satisfies the tangential no-slip condition is a Fredholm equation of the second kind, and that there is a globally convergent Neumann series, thus an iterative solver will be effective. On the other hand, a BIE which uses the normal flux condition on the boundary (and vortex strength distributions) is a Fredholm equation of the first kind, and can lead to ill-conditioned systems of equations [184].

An alternative to using boundary element methods (I think) is to solve the Poisson equation on a grid allowing for discontinuities in field properties. This is similar to what is described in the section on flow around solid boundaries. Deng *et al* [211] describe a method for solving 3D elliptical equations with immersed interfaces.

These are all boundary integral methods, similar to those required for the solution of flow over solid objects.

At high Froude number, the free surface does not deform at all, though, owing to viscosity, it may support slip flow.

Early solutions to this equation were done by matrix-inversion techniques [301], but Baker [100, 297] and Tryggvason [291, 96] used iterative methods.

I am uncertain of the method used in Pullin [302].

The equations for vortex sheet strength in 2D [303, 96, 296, 304], for dipole strength in 2D [303], and vortex strength in 3D [197, 88, 89] involve an acceleration term that is not required in the Boussinesq approximation for weakly-stratified fluids. This acceleration term makes the problem fully dynamical, and requires approximation and discretization in the time domain. This is most often done with a first-order backward difference. Note that the evolution equation for dipole strength includes only first order derivatives on the surface, which is why it is favored by Baker [297].

Zaroodny and Greenberg [305] seem like the first foray into modeling a free surface (and the bottom boundary) as 2D vortex sheets. A direct Biot-Savart integration method is used to solve, and the wave is not allowed to break. Periodic, inviscid, irrotational, incompressible, and non-breaking waves are simulated.

Later, Zalosh [21] calculated vortex sheet motion in two dimensions with surface tension and arbitrary stratification. This used a direct summation approach, but did not model the bottom boundary.

Longuet-Higgins and Cokelet [301, 306] computed the motion of breaking waves, but forced the wave by using an asymmetric pressure field.

Direct simulations were performed by Baker *et al* [100] on the Rayleigh-Taylor instability using 2D vortex sheets with arbitrary stratification. The method requires iterative solution for the fluid acceleration. This solution is equated to a boundary integral method using dipole source distributions [297]. This is because the sheet strength is related to the arclength derivative of the dipole strength.

Baker [297] uses a 4th-order polynomial extrapolation technique that uses data from four previous time steps, but other authors [303] show that a 2nd order R-K method is sufficient.

Pullin [302] solved for the motion of K-H and R-T instability with a vortex sheet with and without surface tension. The formulas are complex.

Tryggvason and Aref introduced an iteration method to solve these flows in [291] and later in [96].

Zufiria [296] uses a unique method to solve the vorticity evolution equation. Upon recognition that the $A = 0$ case involves shock-like behavior of the vortex sheet strength, a Gudonov-scheme is applied in the equations, instead of a standard centered difference operator. The stabilizing effect allows a sharp vortex strength jump to be maintained—a critical feature of the inviscid free-surface Rayleigh-Taylor instability. This method is 1st order in

time and space.

Kerr [303] notes that the error found running their own RTI at variable A was caused exclusively by the centered difference. They instead use a cubic spline to calculate the derivatives on the surface.

Brecht and Ferrante [202] use a 3D VIC particle method with strong stratification, but the equations for vorticity evolution appear different than those reported above—they do not solve a Fredholm equation of the second kind. The rise of one and two bubbles at $0.5 < A < 1.0$ is computed using this method. Their later work [203] presents equations for strong stratification that replace the Boussinesq approximation term as follows:

$$\frac{\nabla\rho}{\rho_0} \times \boldsymbol{g} \rightarrow \nabla(\ln\rho) \times \Big(\boldsymbol{g} - \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\Big)\mathrm{d}V = |\boldsymbol{n}| \ln(\frac{\rho_2}{\rho_1})\, \hat{\boldsymbol{n}}_i \times \Big(\boldsymbol{g} - \frac{\mathrm{d}\boldsymbol{u}_i}{\mathrm{d}t}\Big). \quad (3.19)$$

The authors had to evaluate of the area $|\boldsymbol{n}|$ and normal $\hat{\boldsymbol{n}}_i$ from a spatial distribution of unremeshed particles—a procedure that loses accuracy as the particles separate and wind up around each other.

Simulations of a 2D vortex ring impinging on a density interface appear in Dahm, Scheil and Tryggvason [300] with both strong and weak density jumps and in [307] for both density jumps and free surfaces (and tentatively in 3D).

A two-dimensional vortex method is used by Chen and Vorus [308] to solve for the motion of a free surface with a submerged circular cylinder.

Baroclinic generation of vorticity is done in a three-dimensional vortex sheet method by Knio and Ghoniem [86] with rectangular transport elements containing descriptions of the density jump. The simulations were performed for small Richardson numbers.

$$Ri = \frac{g\,\Delta\rho\,\Delta l}{\rho_0\,U^2} \quad (3.20)$$

In such cases, fluid acceleration is much higher than gravitational acceleration, and the baroclinic source term does not contain $\boldsymbol{g}$, and as such can be written in terms of the fluid acceleration only. A first-order backward finite difference method is used to determine the accelerations in this work. The work performs simulations on a flow with density ratio 2, and also mentions that the method was created for flows with zero Froude number.

$$Fr_1^2 = Fr_2 = \frac{u^2}{gL} \quad (3.21)$$

Rood [309] explains the procedure by which vortex lines reorganize to become normal to a free surface.

Dommermuth [310] studies spurious high-frequency errors in numerical simulations of free-surface flows.

Chang *et al* [119] used a level set method in 2D to compute the motion of interfaces tracking large density jumps. The solution method is pure Euler, though.

Zhang and Ghoniem [311] use a vortex method to compute large density discontinuities in the axisymmetric case.

Scorpio [284] used FMM to accelerate a BEM calculation for a free-surface problem, but not a vortex method.

Lozano *et al* [88] uses a desingularized Biot-Savart integration to solve for strong stratification across a vortex sheet. It also computes the derivatives on the sheet using local coordinates, and also employs a split forward integration scheme.

Haroldsen and Meiron [58] uses a desingularized Biot-Savart integration to solve the motion of a vortex sheet defining a free surface in a doubly-periodic domain in 3D.

A 2D vortex sheet method is used by Reinaud *et al* [207] to simulate a variable-density mixing layer. It is interesting because of the absence of gravity in the simulation—all baroclinic effects are produced by the local acceleration, which is calculated using first and second-order backward difference schemes.

Kotelnikov and Zabusky [137] track particles with circulation in their simulation of a 2D twice-accelerated sine wave, but in an incompressible sense.

Hou *et al* [312] study boundary integral methods in 2D for all sorts of problems: strong stratification, surface tension, and Hele-Shaw flow. An Eulerian method is used by Ye *et al* [214] for bubble dynamics.

Young [313] proposes a method to use LES and a level-set approach to model the interaction of turbulence and a free surface. Simulations of the free-surface 2D Rayleigh-Taylor instability are presented.

Albuquerque and Cottet [223] present a method that couples a finite difference and integral formulas to create an iterative method for solving free-surface problems in 2D. Song and Sirviente propose a FD method for 2D breaking waves [314]. Lörstad, *et al* [315] compare volume-of-fluid (VOF) and immersed boundary (IB) methods, both relying on Eulerian Navier-Stokes solvers, for multifluid flows. Esmaeeli and Tryggvason [316] demonstrate a

3D front-tracking method for multi-fluid flow with phase change. A-Rawahi and Tryggvason [317] demonstrate a 3D front-tracking method for dendritic solidification. Wang and Khoo [106] use a BEM to determine the motion of an multi-fluid interface for simulations of bubbles underwater. De Sousa *et al* [318] use an Eulerian N-S solver with a front-tracking method to compute multi-fluid flows in 3D. Baker and Beale [221] computes the 2D motion of an interface in strongly-stratified flow with a vortex blob method. Shen and Evans [319] present an Eulerian velocity-vorticity method for free-surface and density-stratified flow in 2.5D (an elliptic equation is only solved in the horizontal directions).

3.6 Surface tension

The Weber number reveals the importance of surface tension in a fluid system. The coefficient on the surface tension term in the vorticity evolution equation is We^{-1}.

Zalosh introduced surface tension to the calculation of vortex sheet motion in two dimensions in 1976 [21]. Traditional approaches introduce stability constraints that "stiffen" the system of equations, especially where grid points or Lagrangian points are irregularly-spaced.

An advantage of level-set methods [115] is that the curvature and normal can be calculated on the grid, allowing for smoother interpolation of curvature than a C^1 continuous sheet (piecewise flat triangulated mesh) allows. This same advantage can be achieved in hybrid Eulerian- Lagrangian schemes, where the element-wise scalar gradient is interpolated onto the grid. (As we do.) After the indicator function has been created, one can compute the surface normal from the gradient of the indicator function. Likewise, the curvature is the negative divergence of the normalized surface normal field.

Several works [197, 88, 89, 304] present the formulas of motion for periodic vortex sheets between fluids of different densities with surface tension. Hou, Lowengrub and Shelley present accurate calculations of a periodic vortex sheet in two dimensions with surface tension, and with [295] and without [320] density jump. This also references earlier works where numerical problems with surface tension were identified.

Lozano *et al* [88] mentions that inclusion of surface tension had no effect on the simulations, the reason being that the Weber number was $O(10^5)$ while other terms were $O(1)$.

Torres and Brackbill [104] introduce the point-set method for front-tracking without connectivity. Simulations and formulas for droplet oscillation are included. It is noted that in the 3D case, surface area is the least accurate calculation, making this method inappropriate for vortex sheet calculations (where vorticity is directly related to element area). The work does address parasitic currents that are formed by application of surface tension forces from an interface to a grid. An improvement to the basic projection technique is presented, but requires three Poisson solutions.

Hou *et al* [312] study boundary integral methods in 2D for all sorts of problems: strong stratification, surface tension, and Hele-Shaw flow.

Young [313] also talks about the problem with applying surface tension forces in a level-set method and proposes applying them on the surface elements themselves.

There exists an element-wise method for conservative 3D surface tension on triangle meshes [101, 105]. It was also used in the context of 2D unstructured-grid finite element calculations [321].

Nitsche and Steen [304] present a method for axisymmetric vortex sheets with surface tension.

Popinet and Zaleski [322] present a front-tracking method for surface tension, but in 2D, and in an Eulerian sense.

De Sousa *et al* [318] describe a method for computing surface tension from triangle mesh data based on fitting a plane and a sphere to the mesh.

Brecht and Ferrante [203] suggest that a side effect of the VIC method is that the grid filter introduces a kind of surface tension into the motion of an interface.

3.7 Particle-laden flows

Vortex methods were first used to calculate particle dispersion in 1985 by Crowe, *et al* (Particle Sci. Technol 3, 149, 1985) and 1988 by Chung and Troutt [323], (summary in Crowe, *et al* [324]). These methods assumed no collisions and one-way coupling (the flow affects the particles, but not the reverse.)

Chen and Marshall [13] use FMM and random-walk method in 2D to compute particle-laden flow with 2-way-coupling. In this method, the particles provide a correction to the vorticity transport equation. Though the authors apply this on a particle-wise basis in the FMM calculation, there is

no reason that it cannot be applied on a grid in a VIC calculation. Walther and Koumoutsakos [226] use VIC and PSE in 3D to include large number of solid particles into the flow with 2-way coupling.

Another method for this (and they may be related) is to use impulse vector blobs in addition to vortex blobs [129].

Esmaeeli and Tryggvason [325] provide a method for DNS of deformable particles and Hu [326] contains a summary of methods used for fluid-particle flows. Wallner and Meiburg [327] present a method for two-way particle-laden flows in 2D using a ω-ψ approach.

Squires and Eaton [328] discover via DNS that particles preferentially concentrate in areas of low vorticity and high strain rate, thus possibly invalidating studies of particle-laden flow using uncoupled methods.

3.8 Rotating frame

If the frame of reference is undergoing a rotation at a constant rate $\boldsymbol{\Omega}$, then the vorticity equation can be rewritten in the new frame with addition of two terms. If the total velocity can be written as

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{\Omega} \times \boldsymbol{r} \tag{3.22}$$

then the vorticity equation contains the new terms

$$\boldsymbol{f}_{rotation} = \underbrace{-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})}_{centrifugal force} \underbrace{-2\,(\boldsymbol{\Omega} \times \boldsymbol{v})}_{Coriolis force}. \tag{3.23}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times \boldsymbol{f}_e, \tag{3.24}$$

where $\boldsymbol{v}$ is the fluid velocity in the rotating frame, $\boldsymbol{\Omega}$ is the rotational velocity, and $\boldsymbol{r}$ is a vector from the axis of rotation to the point of evaluation. Note also that

$$-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \Omega^2 \boldsymbol{R} \tag{3.25}$$

where $\boldsymbol{R}$ is the component of the position vector perpendicular to the axis of rotation. See Pope [329], §2.9, for more information.

Kiya and Arie [330] study the Kelvin-Helmholtz instability of a vortex sheet in uniform shear. There are few other vortex sheet calculations related to rotating flows.

Baey *et al* [331] perform an Eulerian calculation for potential vorticity in a planet's atmosphere.

3.9 Transport Elements/Scalar Transport Equation

In this section, we present work on vortex methods that incorporate either co-located source elements for combustion/compressibility calculations, or store other important information on the computational sheet, such as scalar gradients, scalar moments, etc.

The decision to follow this course was driven by the desire to solve the vorticity transport equation along with the scalar transport equation. In many situations, the vorticity is co-located with the scalar gradients.

The choice to co-locate scalar gradient elements with vortex elements is trivial for cases involving two immiscible fluids, but remains logical for fluids systems where viscosity is equivalent to (whatever parameter is used in the scalar convection-diffusion equation in front of the diffusion operator), or when $Sc \approx 1$.

Peters [332] claims that turbulent mixing is dominated by quasi-one-dimensional diffusion layers. This provides motivation for the study of sheets in turbulent mixing flows.

3.9.1 Front-Tracking

Front-tracking in its purest sense (that of adding the computational degree of freedom of the location of a sheet to an Euler solver) was used on the Rayleigh-Taylor in 2D problem by Glimm *et al* [333] and extended to 3D in 1995 (Glimm others, report SUNYSB-AMS-95-17, 1995, "Three dimensional front tracking") and 1998 [102, 103]. Unverdi and Tryggvason use front-tracking for a Navier-Stokes solution to rising viscous bubbles with surface tension [234]. Apparently, they achieve excellent mass conservation with this method.

Front-tracking methods can compute normals and curvatures either element-by-element, or by reconstructing an indicator function and using grid calculations (as discussed below).

Front-tracking does not imply regularization. As proposed by numerous authors, front-tracking is used along with methods for interpolating functions across discontinuities [103] in order to solve problems with dynamic discontinuities in an Eulerian method. So, the next time step is computed using the values on the grid, with the equations modified somewhat by the presence of

the interface. Vortex methods, on the other hand, solve for the system state at subsequent times using *only* the location and strength of the interface. Thus, either a singular integration must be performed over the surface, or regularization must occur.

Front-tracking in 3D shares techniques with automatic 2D triangle meshing methods for finite-difference or finite-element codes [321].

3.9.2 Scalar field definition and reconstruction

As it is computationally more tractable in high-Reynolds number flows to discretize gradients of scalar instead of quantities of scalar, a method to track and recreate the scalar field must be employed. This is nearly the same problem that vortex methods solve, though a scalar field would need to have its gradient be discretized instead. A sample of this procedure is laid out in Ghoniem *et al* [81]. While Ghoniem *et al* [81] relied on assigning vector-valued (temperature) gradients to particles (with no inter-element connectivity) with a method called the "transport element method", later efforts by Dahm and Tryggvason [260, 261] needed only assign scalar-valued gradients to segment elements because connected segments automatically define a direction.

Recreating the scalar field from a collection of Lagrangian gradient elements can be done using any of the methods mentioned in the section on velocity calculation (section 2). These can involve direct integration of the gradient distribution [299, 81], solution of a Laplace equation on a grid [234], or solving the scalar equivalent of the Biot-Savart equation [72]. This last reference contains a thorough description of the math involved.

Knio and Ghoniem [85] track the concentration of two scalars on their Lagrangian surface: oxidizer and fuel concentrations.

3.9.3 Transport element method

Ghoniem *et al* [81] introduce a "transport element method" to update the scalar gradients on Lagrangian elements, thus preventing the need for numerical integration of the scalar gradient transport equation. This method is used in derivative works [72, 86, 85] on rectangular elements in 3D. Ghoniem and Knio [334] use the TEM to compute combustion across a doubly-periodic shear layer. Zhang and Ghoniem [311] applied this method in an axisymmetric case to study the rise of a buoyant cloud. A 2D vortex sheet method

is used by Reinaud *et al* [207] to simulate a variable-density mixing layer, though it uses 73 isopycnic lines to represent the layer (and recognizes that no remeshing can be done normal to the lines). Ghoniem *et al* [81] even adds a core-spreading third fractional step to account for diffusion of temperature gradient.

3.9.4 Local Integral Moment (LIM)

Described in Tryggvason and Dahm [260] and Dahm [335], this method consists of tracking and solving the quasi-1-dimensional diffusion equation on convecting surfaces in turbulent flows. Examples use LIM to track mixing and combustion [261], and vorticity diffusion [159].

And also by Han [336] in reference to the same LIM technology.

A 1-D representation of LIM is described in section III of Beigie, Leonard, and Wiggins [262].

Both LIM and the transport element method are basic extensions of "active interface" problems, as defined by Aref and Tryggvason [292]. They each attach different parameters to the Lagrangian elements, each of which then feeds back into the dynamics of the flow and the interface itself.

3.9.5 Level Set

The level set method describes a scheme for defining a front within a grid, where the front has smooth surface properties. The level set is a scalar-valued "indicator function" and delineates volumes of each fluid. The level set is advected, not the explicit front, and corrected at each step to sharpen its interface. This is an example of a "front capturing" technique, which is different from a "front tracking" technique in that the latter maintains an explicit representation of the interface. Level sets are amenable to grid solvers, hence their popularity.

After the indicator function has been created, one can compute the surface normal from the gradient of the indicator function. Likewise, the curvature is the negative divergence of the normalized surface normal field.

Only recently [124] have level set methods been used to track material quantities on propagating interfaces. This is the first step in their use as true transport elements.

3.9.6 Combustion

Ghoniem and Knio [337] use a two-dimensional vortex-particle core-spreading method with transport elements to study shear flow-combustion interactions with single-step, Arrhenius kinetics. The actual methods is described in [81]. Chang *et al* [261] used locally 1D self-similar shape functions to simulate mixing and combustion on a Kelvin- Helmholtz instability in 2D. Knio and Ghoniem [85] compute the chemically-reacting shear layer under the limitation of infinite-rate kinetics and infinitely-small heat release. Combustion is treated in two dimensional vortex simulations by Soteriou and Ghoniem [338], who use the transport element method and a volumetric expansion term to model exothermic combustion for finite and infinite-rate kinetics. Najm *et al* [43] use a combined Lagrangian-Eulerian method for reacting flow with compressibility effects. Lakkis and Ghoniem [272] develop a particle method for calculating radiative transport in a non-scattering medium—an effect that is present in many combustion experiments. Later, those same authors presented an axisymmetric grid-free combustion vortex method [339].

3.9.7 Fractal representation

In under-resolved simulations using explicit front-tracking methods, the subfilter-scale complexity of the true interface must be accounted for.

Jiménez and Martel [340] studies the fractal dimension of a 2-D mixing layer.

Chorin [78] uses a vortex filament simulation to estimate the Hausdorff dimension of vorticity in developed turbulence.

3.10 Compressibility Effects

Many of the assumptions used in this report become inaccurate when describing flow in the compressible regime (generally taken to be $M \geq 0.3$). The Lagrangian techniques that we have employed, though, are no less valid. In addition to requiring a full description of the vorticity field, a compressible vortex method also needs to describe the dilatation field. Dilatation can be caused by processes in incompressible flows such as low-heat-release combustion.

Mas-Gallic *et al* [341] present a 2D method that uses Lagrangian particles

to solve the convection equations for vorticity and density while using finite-difference schemes for the remaining equations.

Glimm *et al* [102, 103] has used front-tracking techniques along with traditional grid-based solvers to track shocks and compute compressible flows. These are not vortex methods, though.

Najm *et al* [43] also use a combined Lagrangian-Eulerian method for reacting and compressible flow. Gharakhani and Ghoniem [342] compute the flow in a combustion chamber bound by a moving cylinder by applying a uniform source field over the volume, which is a precursor to full grid-free compressible simulations. Axisymmetric grid-free combustion was achieved by Lakkis and Ghoniem [339].

3.10.1 Aerodynamic Sound

Very slightly compressible flows allow the splitting of the compressible part of the equations from the Navier-Stokes part. Then, Lighthill's acoustic pressure equation can be used to produce a differential equation in time for the pressure field at a point in the irrotational far-field. This is shown in Pothou [74], who uses a vortex filament method to predict the acoustic field resulting from the impact of two vortex rings.

3.10.2 Co-location with source particles

Looking at the Helmholtz decomposition, this seems reasonable.

This was mentioned in some Ghoniem paper, I thought.

Eldredge *et al*, in [343, 344], present a dilating particle vortex method, whereby particles carry vorticity, dilatation, enthalpy, entropy, and density. Simulations in two dimensions are presented.

Particles in Nitsche and Strickland [345] carry vorticity, divergence, temperature, and density.

Thirifay and Winckelmans [346] use particles that carry vorticity, temperature, density, and species density in their simulation of a reacting diffusion flame in 2D.

Chapter 4

Sample Simulations

Vortex methods have been used to study a great many canonical fluid dynamics problems. In addition to references to those, we shall mention many studies of fluid phenomena which have not had vortex methods applied to them.

4.1 Free-boundary, homogenous flows

The most basic test of a vortex method is the Perlman test [63], which defines a tight vortex spot, and is a common test of 2D vortex codes [41, 109]. Similar are simulations of elliptical vortex patches [347]. In this category are also simulations of more than one circular vortex patch [156] and many others.

A single shear layer simulated in two dimensions was the first and most popular early test of vortex methods [16, 348, 150, 349, 98]. It is known that the most unstable mode in this, the Kelvin-Helmholtz roll-up is at $\lambda = 13.2\,\sigma$ [350, 81]. Its stability has been addressed [151], and desingularization has been proposed [26]. Shear layers with a Gaussian distribution across the layer are a favorite of Ghoniem [81] and others. A two-dimensional shear layer is analyzed with a three-dimensional method in [89, 244]. Vortex methods are compared to Eulerian methods in [159]. The Kelvin-Helmholtz roll-up is also a popular test problem for Eulerian methods [141].

Its analogue in three dimensions is the doubly-periodic shear layer, which has likewise been used to study flow [159, 118, 89, 244], entrainment [72, 86], and combustion [334]. Experimental results are presented for the formation of streamwise vortices [351] and the growth of large scales in the plane mixing

layer [352], and computational results are presented for secondary instability [353], transition to turbulence [354] and spanwise scale selection [355].

Putting two shear layers of opposite-signed vorticity together makes a more accurate representation of a plane mixing layer [18, 66], also 2-D. The simulation of two sheets of opposite-signed vorticity generates the families of Kármán vortex streets. Tracking this instability from an initially flat splitter plate creates a space-developing mixing layer [356] (2-D, also studied turbulent fluctuations in the mixing layer) and [255] (2D VIC, also includes turbulence statistics in the mixing layer).

Another related simulation has two sinusoidally-perturbed shear layers of similar-signed vorticity spaced evenly in a doubly-periodic area (2D). This problem has been tackled by Eulerian ω-ψ schemes [141, 357]. Test cases for three dimensional vortex methods are listed in [69], §3, and include Stuart's family of periodic shear layers [358].

A shear layer emanating from a semi-infinite flat plate rolls up [359, 201, 161] according to a similarity solution.

Vortex rings are commonly studied, and appear in both thick [71, 48, 178] and thin [39] varieties, perturbed [71], head-on and head-off pair collision [170, 39], leapfrogging [8, 178], creation from a disc of vorticity [360, 82], oblique merging pairs [361, 44, 112, 82], sets of four [69], in axisymmetric flow with diffusion and combustion [339], only diffusion [256, 110], and with neither [362]. A good analysis of viscous vortex rings appears in Saffman [3], which mentioned Hill's spherical vortex and references Fraenkel's work on steady, thin-cored vortex rings [363] and Norbury's steady, thick-cored rings [364]. Tung and Ting [23] and Saffman [24] found that the distribution of vorticity across the core of a viscous vortex ring with small cross-section is Gaussian. The decay of a viscous vortex pair is studied in 2D using a heuristic method [365] and in an axisymmetric vortex method [110]. There are many examples of experimental work on vortex rings [366]. Dabiri and Gharib [367] demonstrate the near-term scaling of viscous vortex rings including the effects of fluid entrainment. A complete summary appears in Shariff [368]. Finally, Maxworthy [369] showed that a vortex ring becomes turbulent when the Reynolds number (Γ/ν) is greater than about 600.

Similarly to the flow over a sphere, one can run a simulation of a spherical vortex sheet with an initial vorticity distribution equivalent to that required for potential flow past a co-located sphere [370, 44, 371, 89, 154, 158] or cylinder [27, 370, 87, 154] in a unit freestream. These simulations result in a traveling vortex pair or ring.

A simple extension of a vortex ring is the circular starting jet (the spatially-growing jet) [372, 373], and the periodic jet (the temporally-growing jet), with swirl [70] or without [374, 67, 87]. A periodically-forced jet has been shown to entrain nearly twice as much quiescent fluid (Bremhorst, JFluidsEng 103, 605, 1981). A jet excited axially and azimuthally, can produce a bifurcating round jet [375]. Experiments and analysis appear in [376, 377]. Coaxial jets are experimentally studied in [378]. Computational and experimental results for both buoyant and non-buoyant jets in weak crossflow are presented in Yuan and Street [379] and computational results and scaling relationships by your buddy Javier (no citation yet). Square jets are simulated with N-S solvers [380].

A turbulent spot is simulated by Leonard [381].

Vortex breakdown is where a vortex tube with axial vorticity has a kink in it [69]. A vortex tube with swirl flow is a simple 3D vortex sheet flow [80].

The motion of a surface with a fixed initial vortex strength is studied in 2D [201, 382, 383] for the roll-up of an aircraft wake. Related to that is the roll-up of a vortex disc in 3D with azimuthal perturbations, done by Lindsay and Krasny [82]

A columnar vortex reacts to a vortex loop [384].

A final common vortex method test is that of a doubly-periodic domain in two dimensions filled with vorticity which exhibits a k^{-1} decay in Fourier space and random phase [281].

I don't know what a Taylor-Green problem is, but an early 3D vortex method [385] simulates it.

4.2 Multifluid flows

A density interface that undergoes passage of a shock represents a Richtmyer-Meshkov instability, and can be simulated using front-tracking methods in 2D for planar [102, 103] or curved [386] geometries. Growth rates are given for 2D in [387]. Shocked density interface experiments [388] are also computationally tractable. None of these calculations evolve strengths on the interface, and are thus computationally different from vortex methods. Except for Kotelnikov and Zabusky [137], who track particles with circulation corresponding to a 2D twice-accelerated sine wave, but in an incompressible sense.

The two-dimensional Kelvin-Helmholtz instability has been studied ex-

tensively with vortex methods, both for flow [26, 89, 244], mixing [261], variable-density [207], free-surface flow [297], surface tension [295, 320, 312], and combustion [337, 261], and with level set methods [118, 122].

A turbulent ring is shot into a stratified interface in the experiments done by Linden [389], and the same for a jet by Shy and Breidenthal [390]. A laminar vortex ring is shot into density interfaces and free surfaces in 2D in [391, 392, 393, 300, 307] and in 3D in [307, 244].

Physical studies and rates of spread for mixing layers of various density ratios can be found in Brown and Roshko [394]. Lozano *et al* [88] simulates development of a liquid sheet up to breakup using 3D vortex sheet methods. A particle-laden mixing layer is simulated in 2D [327]. Growth rates are studied in 2D and 3D woth vortex methods in [244].

This same layer without initial vorticity but with an unstable density interface is called a Rayleigh-Taylor instability. It was computed with vortex methods in 2D [100, 303, 96, 296, 221] and 3D (not yet), and by Euler-front-tracking methods in 2D [333, 395] and 3D for immiscible [396] and miscible [397] fluids. A good introduction to the problem is given in Tryggvason [96]. The initial growth of this layer follows linear theory, recounted in [398], and the growth rates in a rotating environment are given in [399].. That same sinusoidally-perturbed layer with a stable density interface oscillates with a period predicted by classical theory for infinitesimal perturbations [21].

The effects of buoyancy are also addressed in the simulation of a buoyant cylinder [56, 299, 157] or sphere [202, 203, 283, 89, 244]. These conditions produce vortex pairs and rings, and scaling relations exist for the starting buoyant plume [400, 401] and thermal [402, 400], and for the axisymmetric regular [362] and buoyant vortex rings [362, 400, 283]. Numerical studies have been performed of the motion of a buoyant vortex pair (2D) in a stably stratified atmosphere [398]. In one of these ([283]), the authors study microbursts, whereby an inverse thermal impacts the ground. The rise of plane bubbles is studied in Birkhoff and Carter in J Math. Mech. 6, 769-779, 1957. Experiments are common in earlier work: turbulent thermals [402]. Eulerian methods are frequently used for bubble dynamics [403, 214].

Free surface calculations are like multifluid calculations but with an Atwood number of 1 (density ratio is infinite). Breaking waves are simulated in 2D [297, 301, 14]. Gravity currents are just breaking waves in the Boussinesq limit. See Benjamin [404] for discussion, and [56] for simulations. These are also called "weather fronts." Haroldsen and Meiron [58] simulate the simple (2D) Stokes wave in their 3D free-surface point vortex method, and go on

to simulate a 3D wave whose initial shape is a Gaussian. A buoyant sphere and torus with a massless core (free-surface equivalent) are simulated by Lundgren and Mansour [405].

The Taylor instability of a thin fluid layer is numerically simulated in Verdon *et al* [406].

An interesting instability is the Batchelor-Nitsche [407] instability, which consists of a periodically stratified expanse of fluid. It was later studied by Proctor [408].

The most common test of surface tension effects is the radial oscillation of a bubble in the free-surface (Rayleigh-Plesset equation) [409] or non-free-surface regimes.

4.3 Flows with solid boundaries

Internal flows are of the more common fluids solutions involving solid boundaries. The most common of these is that of a driven-cavity. To my knowledge, this has not been attempted with vortex methods, though many Eulerian calculations exist in two [138, 141] and three [138] dimensions.

There are any number of standard simulations for flow over a solid object. 2D Flow over a cylinder [218, 228, 177, 192, 239, 53, 188, 49] and 3D flow over a sphere [410, 113, 37] are common simulations. Vortex ring impaction on a solid surface appears frequently in two [411] and three [192, 412] dimensions, as does vortex ring impaction on a cylinder (3D) [49] and vortex ring impaction on a blade [38]. Experimental results for the above situation appear in [413].

Dynamic boundaries are most commonly done by simulating the flow over an oscillating cylinder or wing section [200, 50]. The dynamic flow around a rotating and pitching marine propeller is solved using BEM and vortex methods by Politis [83]. There are very few simulations of dynamic 3D boundaries, but if they were done, things like a spinning sphere, or flapping plate, or swimming fish or flying bird or insect would be excellent.

Flexible boundary flows are less common in the literature, despite their ubiquity in nature. Studying the flow over heart valves seems to be a good funding idea [208, 233]. Flow around swimming fish and birds is likely next.

Actual physical experiments are necessary for some flows. Oddie *et al* [414] tests oil-water and oil-water-gas mixtures in horizontal and inclined pipes. Issa [415] shows how a 1-D model can reasonably predict the statistics

of slug flow in 2-fluid pipes.

Numerically similar to the equations used for baroclinic generation of vorticity and the vortex sheet's subsequent motion are the equations for flow of a highly-viscous fluid through a porous medium. The Rayleigh-Taylor instability in vortex flows becomes the Taylor-Saffman instability. The resulting flow is similar to a space-filling diffusion-limited aggregation flow. This is simulated in [56, 295, 312]. The general flow is called Hele-Shaw flow and is demonstrated numerically in [56, 291, 292, 294, 312].

4.4 Compressible flows

Two-dimensional schemes for flows with heat release and its accompanying velocity divergence have been proposed in either the fully-Lagrangian or Lagrangian-Eulerian senses [43].

4.5 Closing remarks

As shown above, there are many variations on the classical vortex method initially proposed by Rosenhead. We hope that we can add to that body of knowledge.

Bibliography

[1] C. G. Speziale, On the advantages of the vorticity-velocity formulation of the equations of fluid dynamics, J. Comput. Phys. 73 (1987) 476–480.

[2] H. Poincaré, Théorie des Tourbillons, Gauthier-Villars, 1893.

[3] P. G. Saffman, Vortex Dynamics, Cambridge Univ. Press, Cambridge, UK, 1992.

[4] H. J. Lugt, Vortex Flow in Nature and Technology, Krieger Publishing Company, Malabar, FL, USA, 1983.

[5] T. Schwenk, Sensitive Chaos: The Creation of Flowing Forms in Water and Air, Rudolf Steiner Press, London, UK, 1965.

[6] L. Kelvin, On vortex atoms, Phil. Mag. 34 (1867) 15–24.

[7] F. H. Harlow, J. E. Welch, Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface, Phys. Fluids 8 (12) (1965) 2182–2189.

[8] J. Steinhoff, D. Underhill, Modification of the Euler equations for "vorticity confinement": Application to the computation of interacting vortex rings, Phys. Fluids 6 (8) (1994) 2738–2743.

[9] G.-H. Cottet, P. Koumoutsakos, M. L. Ould-Salihi, Vortex methods with spatially varying cores, J. Comput. Phys. 162 (2000) 164–185.

[10] G.-H. Cottet, B. Michaux, S. Ossia, G. VanderLinden, A comaprison of spectral and vortex methods in three-dimensional incompressible flows, J. Comput. Phys. 175 (2002) 702–712.

[11] M. L. Ould-Salihi, G.-H. Cottet, M. E. Hamraoui, Blending finite-difference and vortex methods for incompressible flow computations, SIAM J. Sci. Comput. 22 (5) (2000) 1655–1674.

[12] A. Leonard, Vortex methods for flow simulation, J. Comput. Phys. 37 (1980) 289–335.

[13] H. Chen, J. S. Marshall, A Lagrangian vorticity method for two-phase particulate flows with two-way phase coupling, J. Comput. Phys. 148 (1999) 169–198.

[14] J. J. Monaghan, Simulating free surface flows with SPH, J. Comput. Phys. 110 (1994) 399–406.

[15] L. Rosenhead, The spread of vorticity in the wake behind a cylinder, Proc. Roy. Soc. London Ser. A 127 (1930) 590–612.

[16] L. Rosenhead, The formation of vorticies from a surface of discontinuity, Proc. Roy. Soc. London Ser. A 134 (1931) 170–192.

[17] G. Birkhoff, J. Fisher, Do vortex sheets roll up?, Rend. Circ. Math. Palermo, Ser. 2 8 (1959) 77–90.

[18] F. H. Abernathy, R. E. Kronauer, The formation of vortex streets, J. Fluid Mech. 13 (1962) 1–20.

[19] A. J. Chorin, P. S. Bernard, Discretization of a vortex sheet, with an example of roll-up, J. Comput. Phys. 13 (3) (1973) 423–429.

[20] K. Kuwahara, H. Takami, Numerical studies of two-dimensional vortex motion by a system of point vortices, J. Physical Society of Japan 34 (1) (1973) 247–253.

[21] R. G. Zalosh, Discretized simulation of vortex sheet evolution with buoyancy and surface tension effects, AIAA Journal 14 (11) (1976) 1517–1523.

[22] B. Couët, O. Buneman, A. Leonard, Simulation of three-dimensional incompressible flows with a vortex-in-cell method, J. Comput. Phys. 39 (2) (1981) 305–328.

[23] C. Tung, L. Ting, Motion and decay of a vortex ring, Phys. Fluids 10 (5) (1967) 901–910.

[24] P. G. Saffman, Approach of a vortex pair to a rigid free surface in viscous fluid, Studies in Applied Mathematics 49 (4) (1970) 371–380.

[25] Y. Nakamura, A. Leonard, P. Spalart, Vortex simulation of an inviscid shear layer, in: AIAA/ASME 3rd Joint Thermophysics, Fluids, Plasma and Heat Transfer Conference Conference, no. AIAA-82-0948, 1982.

[26] R. Krasny, Desingularization of periodic vortex sheet roll-up, J. Comput. Phys. 65 (1986) 292–313.

[27] J. W. Rottman, P. K. Stansby, On the 'δ-equations' for vortex sheet evolution, J. Fluid Mech. 247 (1993) 527–549.

[28] A. J. Chorin, Vortex models and boundary layer instability, SIAM J. Sci. Stat. Comput. 1 (1980) 1–21.

[29] J. T. Beale, A. Majda, Vortex methods I: Convergence in three dimensions, Math. Comput. 29 (159) (1982) 1–27.

[30] O. H. Hald, V. D. Prete, Convergence of vortex methods for Euler's equations, Math. Comput. 32 (1978) 791–809.

[31] O. H. Hald, The convergence of vortex methods, II, SIAM J. Numer. Anal. 16 (5) (1979) 726–755.

[32] J. T. Beale, A. Majda, Vortex methods II: Higher order accuracy in two and three dimensions, Math. Comput. 29 (159) (1982) 29–52.

[33] J. T. Beale, A convergent 3-D vortex method with grid-free stretching, Math. Comput. 46 (174) (1986) 401–424.

[34] J. T. Beale, On the accuracy of vortex methods at large times, in: IMA Workshop on Computational Fluid Dynamics and Reacting Gas Flows, Springer-Verlag, 1988, p. 19.

[35] G. S. Winckelmans, A. Leonard, Improved vortex methods for three-dimensional flows, in: R. E. Caflisch (Ed.), Proceedings of the Workshop on Mathematical Aspect of Vortex Dynamics, Leesburg, Virginia, April 25-27, 1988, 1989, pp. 25–35.

[36] G. S. Winckelmans, A. Leonard, Contributions to vortex particle methods for the computation of three-dimensional incompressible unsteady flows, J. Comput. Phys. 109 (1993) 247–273.

[37] P. Ploumhans, G. S. Winckelmans, J. K. Salmon, A. Leonard, M. S. Warren, Vortex methods for direct numerical simulation of three-dimensional bluff body flows: Application to the sphere at Re=300, 500, and 1000, J. Comput. Phys. 178 (2002) 427–463.

[38] J. S. Marshall, J. R. Grant, Penetration of a blade into a vortex core: vorticity response and unsteady blade forces, J. Fluid Mech. 306 (1996) 83–109.

[39] C. H. Liu, D. J. Doorly, Vortex particle-in-cell method for three-dimensional viscous unbounded flow computations, Intl. J. Num. Meth. Fluids 32 (2000) 29–50.

[40] H. N. Najm, A hybrid vortex method with deterministic diffusion, in: J. T. B. et al. (Ed.), Vortex Flows and Related Numerical Methods, Kluwer Academic Publishers, 1993, pp. 207–222.

[41] J. A. Strain, 2D vortex methods and singular quadrature rules, J. Comput. Phys. 124 (1996) 131–145.

[42] H. O. Nordmark, Rezoning for higher order vortex methods, J. Comput. Phys. 97 (1991) 366–397.

[43] H. N. Najm, R. B. Milne, K. D. Devine, S. N. Kempa, A coupled Lagrangian-Eulerian scheme for reacting flow modeling, ESAIM Proc. 7 (1999) 304–313.

[44] G. S. Winckelmans, J. K. Salmon, M. S. Warren, A. Leonard, B. Jodoin, Application of fast parallel and sequential tree codes to computing three-dimensional flows with the vortex element and boundary element methods, ESAIM Proc. 1 (1996) 225–240.

[45] P. Koumoutsakos, A. Leonard, Direct numerical simulations using vortex methods, in: Goule, Cottet, Huberson (Eds.), Proc. NATO Adv. Res. Workshop: Vortex Flows and Related Numerical Methods, Kluwer, 1992, pp. 179–190.

[46] G.-H. Cottet, M. L. Ould-Salihi, M. E. Hamraoui, Multi-purpose regridding in vortex methods, ESAIM Proc. 7 (1999) 94–103.

[47] L. A. Barba, A. Leonard, C. B. Allen, Numerical investigations on the accuracy of vortex methods with and without remeshing, in: 16th AIAA Computational Fluid Dynamics Conference, Orlando, FL, 2003.

[48] J. R. Mansfield, O. M. Knio, C. Meneveau, Towards Lagrangian large vortex simulation, ESAIM Proc. 1 (1996) 49–64.

[49] G.-H. Cottet, P. Poncet, Advances in direct numerical simulations of 3D wall-bounded flows by Vortex-in-Cell methods, J. Comput. Phys. 193 (2003) 136–158.

[50] J. D. Eldredge, Effecient tools for the simulation of flapping wing flows, in: 43rd Aerospace Sciences Meeting, AIAA, 2005.

[51] L. F. Rossi, Merging computational elements in vortex simulations, SIAM J. Sci. Comput. 18 (4) (1997) 1014–1027.

[52] G. Daeninck, P. Ploumhans, G. S. Winckelmans, Simulation of three-dimensional bluff-body flows using vortex methods: from direct numerical simulation towards large-eddy simulation modelling, J. Turb. 3 (043).

[53] P. Ploumhans, G. S. Winckelmans, Vortex methods for high-resolution simulations of viscous flows past bluff body of general geometry, J. Comput. Phys. 165 (2000) 354–406.

[54] P. Moeleker, A. Leonard, Lagrangian methods for the tensor-diffusivity subgrid model, J. Comput. Phys. 167 (2001) 1–21.

[55] P. Chatelain, A. Leonard, Face-centred cubic lattices and particle redistribution in vortex methods, J. Turb. 3 (046).

[56] J. C. S. Meng, J. A. L. Thomson, Numerical studies of some nonlinear hydrodynamic problems by discrete vortex element methods, J. Fluid Mech. 84 (3) (1978) 433–453.

[57] D. Fishelov, Vortex methods for slightly viscous three-dimensional flow, SIAM J. Sci. Stat. Comput. 11 (3) (1990) 339–424.

[58] D. J. Haroldsen, D. I. Meiron, Numerical calculation of three-dimensional interfacial potential flows using the point vortex method, SIAM J. Sci. Comput. 20 (1998) 648–683.

[59] J. Goodman, T. Y. Hou, J. S. Lowengrub, Convergence of the point vortex method for the 2-D Euler equations, Communications on Pure and Applied Mathematics 43 (1990) 415–430.

[60] T. Y. Hou, J. S. Lowengrub, Convergence of the point vortex method for the 3-D Euler equations, Communications on Pure and Applied Mathematics 43 (1990) 965–981.

[61] G.-H. Cottet, J. Goodman, T. Y. Hou, Convergence of the grid-free point vortex method for the three-dimensional Euler equations, SIAM J. Numer. Anal. 28 (2) (1991) 291–307.

[62] T. Y. Hou, J. S. Lowengrub, R. Krasny, Convergence of a point vortex method for vortex sheets, SIAM J. Numer. Anal. 28 (2) (1991) 308–320.

[63] M. Perlman, On the accuracy of vortex methods, J. Comput. Phys. 59 (1985) 200–223.

[64] B. Merriman, Particle approximation, in: C. Anderson, C. Greengard (Eds.), Vortex Dynamics and Vortex Methods, no. 28 in Lectures in Applied Mathematics, American Mathematical Society, 1991, pp. 481–546.

[65] G. K. Batchelor, An Introduction to Fluid Dynamics, Cabridge University Press, 1967, 1973.

[66] W. T. Ashurst, E. Meiburg, Three-dimensional shear layers via vortex dynamics, J. Fluid Mech. 189 (1988) 87–116.

[67] J. E. Martin, E. Meiburg, Numerical investigation of three-dimensional evolving jets subject to axisymmetric and azimuthal perturbation, J. Fluid Mech. 230 (1991) 271–318.

[68] A. Leonard, Numerical simulation of interacting, three-dimensional vortex filaments, in: Proceedings of the IV Intl. Conference on Numerical Methods of Fluid Dynamics, no. 35 in Lecture Notes in Physics, Springer-Verlag, 1975, pp. 245–250.

[69] A. Leonard, Computing three-dimensional incompressible flows with vortex elements, Ann. Rev. Fluid Mech. 17 (1985) 523–559.

[70] J. E. Martin, E. Meiburg, Three-dimensional vorticity dynamics in a swirling jet, ESAIM Proc. 1 (1996) 539–552.

[71] O. M. Knio, A. F. Ghoniem, Numerical study of a three-dimensional vortex method, J. Comput. Phys. 86 (1990) 75–106.

[72] O. M. Knio, A. F. Ghoniem, Three-dimensional vortex simulation of rollup and entrainment in a shear layer, J. Comput. Phys. 97 (1991) 172–223.

[73] D. W. Moore, P. G. Saffman, The motion of a vortex filament with axial flow, Phil. Trans. Roy. Soc. London Ser. A 272 (1972) 403–429.

[74] K. P. Pothou, S. G. Voutsinas, S. G. Huberson, O. M. Knio, Application of 3D particle method to the prediction of aerodynamic sound, ESAIM Proc. 1 (1996) 349–362.

[75] R. Klein, O. M. Knio, Improved thin-tube models for slender vortex simulations, J. Fluid Mech. 284 (1995) 275–321.

[76] O. M. Knio, R. Klein, Improved thin-tube models for slender vortex simulations, J. Comput. Phys. 163 (2000) 68–82.

[77] C. Greengard, Convergence of the vortex filament method, Math. Comput. 47 (176) (1986) 387–398.

[78] A. J. Chorin, Estimates of intermittency, spectra, and blow-up in developed turbulence, Communications on Pure and Applied Mathematics 34 (1981) 853–866.

[79] H. L. Berk, K. V. Roberts, The water-bag model, Methods in Computational Physics 9 (1967) 87–134.

[80] T. Sakajo, Numerical computation of a three-dimensional vortex sheet in a swirl flow, Fluid Dynamic Research 28 (2001) 423–448.

[81] A. F. Ghoniem, G. Heidarinejad, A. Krishnan, Numerical simulation of a thermally stratified shear layer using the vortex element method, J. Comput. Phys. 79 (1988) 135–166.

[82] K. Lindsay, R. Krasny, A particle method and adaptive treecode for vortex sheet motion in three-dimensional flow, J. Comput. Phys. 172 (2001) 879–907.

[83] G. K. Politis, Simulation of unsteady motion of a propeller in a fluid including free wake modelling, Engineering Analysis with Boundary Elements 28 (2004) 633–653.

[84] M. E. Agishtein, A. A. Migdal, Dynamics of vortex surfaces in three dimensions: Theory and simulations, Physica D 40 (1989) 91–118.

[85] O. M. Knio, A. F. Ghoniem, Vortex simulation of a three-dimensional reacting shear layer with infinite-rate kinetics, AIAA J. 30 (1) (1992) 105–116.

[86] O. M. Knio, A. F. Ghoniem, The three-dimensional structure of periodic vorticity layers under non-symmetric conditions, J. Fluid Mech. 243 (1992) 353–392.

[87] M. Brady, A. Leonard, D. I. Pullin, Regularized vortex sheet evolution in three dimensions, J. Comput. Phys. 146 (1998) 520–545.

[88] A. Lozano, A. García-Olivares, C. Dopazo, The instability growth leading to a liquid sheet breakup, Phys. Fluids 10 (9) (1998) 2188–2197.

[89] C. Pozrikidis, Theoretical and computational aspects of the self-induced motion of three-dimensional vortex sheets, J. Fluid Mech. 425 (2000) 335–366.

[90] J. T. Beale, Methods for computing singular and nearly singular integrals, J. Turb. 3 (041).

[91] P. S. Bernard, A deterministic vortex sheet method for boundary layer flow, J. Comput. Phys. 117 (1995) 132–145.

[92] P. S. Bernard, A vortex method for wall bounded turbulent flows, ESAIM Proc. 1 (1996) 15–31.

[93] A. Gharakhani, A. F. Ghoniem, Three-dimensional vortex simulation of time dependent incompressible internal viscous flows, J. Comput. Phys. 134 (1997) 75–95.

[94] D. M. Summers, A. J. Chorin, Hybrid vortex/magnet methods for flow over a solid boundary, ESAIM Proc. 1 (1996) 65–76.

[95] R. E. Caflisch, J. S. Lowengrub, Convergence of the vortex method for vortex sheets, SIAM J. Numer. Anal. 26 (5) (1989) 1060–1080.

[96] G. Tryggvason, Numerical simulations of the Rayleigh-Taylor instability, J. Comput. Phys. 75 (1988) 253–282.

[97] D. G. Dritschel, Contour surgery: A topological reconnection scheme for extended integrations using contour dynamics, J. Comput. Phys. 77 (1988) 240–266.

[98] S.-C. Kim, J.-Y. Lee, S.-I. Sohn, Long time computation of two-dimensional vortex sheet by point vortex method, J. Physical Society of Japan 72 (8) (2003) 1968–1976.

[99] S. Kwak, C. Pozrikidis, Adaptive triangulation of evolving, closed, or open surfaces by the advancing-front method, J. Comput. Phys. 145 (1998) 61–88.

[100] G. R. Baker, D. I. Meiron, S. A. Orszag, Vortex simulations of the Rayleigh-Taylor instability, Phys. Fluids 23 (8) (1980) 1485–1490.

[101] G. Tryggvason, B. Bunner, A. Esmaeeli, D. Juric, N. Al-Rawahi, W. Tauber, J. Han, S. Nas, Y.-J. Jan, A front-tracking method for the computations of multiphase flow, J. Comput. Phys. 169 (2001) 708–759.

[102] J. Glimm, M. J. Graham, J. W. Grove, X. L. Li, T. M. .Smith, D. Tan, F. Tangerman, Q. Zhang, Front tracking in two and three dimensions, Computers Math. Applic. 35 (7) (1998) 1–11.

[103] J. Glimm, J. W. Grove, X. L. Li, K.-M. Shyue, Y. Zeng, Q. Zhang, Three-dimensional front tracking, SIAM J. Sci. Comput. 19 (3) (1998) 703–727.

[104] D. J. Torres, J. U. Brackbill, The point-set method: Front-tracking without connectivity, J. Comput. Phys. 165 (2000) 620–644.

[105] S. Shin, D. Juric, Modeling three-dimensional multiphase flow using a level contour reconstruction method for front tracking without connectivity, J. Comput. Phys. 180 (2002) 427–470.

[106] C. Wang, B. C. Khoo, An indirect boundary element method for three-dimensional explosion bubbles, J. Comput. Phys. 194 (2004) 451–480.

[107] E. Aulisa, S. Manservisi, R. Scardovelli, A surface marker algorithm coupled to an area-preserving marker redistribution method for three-dimensional interface tracking, J. Comput. Phys. 197 (2004) 555–584.

[108] G. Russo, J. A. Strain, Fast triangulated vortex methods for the 2D Euler equations, J. Comput. Phys. 111 (1994) 291–323.

[109] J. A. Strain, Fast adaptive vortex methods, J. Comput. Phys. 132 (1997) 108–122.

[110] M. Carley, A triangulated vortex method for the axisymmetric Euler equations, J. Comput. Phys. 180 (2002) 616–641.

[111] S. A. Huyer, J. R. Grant, Solution of two-dimensional vorticity equation on a Lagrangian mesh, AIAA Journal 38 (5) (2000) 774–783.

[112] J. R. Grant, J. S. Marshall, Inviscid interaction of vortex rings: approach to singularity?, ESAIM Proc. 7 (1999) 183–194.

[113] J. S. Marshall, J. R. Grant, A. A. Gossler, S. A. Huyer, Vorticity transport on a Lagrangian tetrahedral mesh, J. Comput. Phys. 161 (2000) 85–113.

[114] J. D. Ramshaw, Conservative rezoning algorithm for generalized two-dimensional meshes, J. Comput. Phys. 59 (1985) 193–199.

[115] S. Osher, J. A. Sethian, Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations, J. Comput. Phys. 79 (1988) 12–49.

[116] S. Osher, R. P. Fedkiw, Level set methods: An overview and some recent results, J. Comput. Phys. 169 (2001) 463–502.

[117] J. A. Sethian, Evolution, implementation, and application of level set and fast marching methods for advancing fronts, J. Comput. Phys. 169 (2001) 597–611.

[118] E. Harabetian, S. Osher, C.-W. Shu, An Eulerian approach for vortex motion using a level set regularization procedure, J. Comput. Phys. 127 (1996) 15–26.

[119] Y. C. Chang, T. Y. Hou, B. Merriman, S. Osher, A level set formulation of Eulerian interface capturing methods for incompressible fluid flows, J. Comput. Phys. 124 (1996) 449–464.

[120] H.-K. Zhao, T. Chan, B. Merriman, S. Osher, A variational level set approach to multiphase motion, J. Comput. Phys. 127 (1996) 179–195.

[121] J. A. Strain, Tree methods for moving interfaces, J. Comput. Phys. 151 (1999) 616–648.

[122] M. Herrmann, An Eulerian level-set/vortex-sheet method for two-phase interface dynamics, in: Annual Research Briefs, Center for Turbulence Research, 2002, pp. 103–114.

[123] D. Enright, R. Fedkiw, J. Ferziger, I. Mitchell, A hybrid particle level set method for improved interface capturing, J. Comput. Phys. 183 (2002) 83–116.

[124] D. Adalsteinsson, J. A. Sethian, Transport and diffusion of material quantities on propagating interfaces via level set methods, J. Comput. Phys. 185 (2003) 271–288.

[125] T. S. Lundgren, Strained spiral vortex model for turbulent fine structure, Phys. Fluids 25 (1982) 2193–2203.

[126] T. F. Buttke, Velicity methods: Lagrangian numerical methods which preserve the Hamiltonian structure of incompressible fluid flow, in: J. T. B. et al. (Ed.), Vortex Flows and Related Numerical Methods, Kluwer Academic Publishers, 1993, pp. 39–57.

[127] N. J. Z. S. M. Zeng, Shock cavity implosion morphologies and vortical projectile generation in axisymmetric shock-spherical fast/slow bubble interactions, J. Fluid Mech. 362 (1998) 327–346.

[128] R. Cortez, On the accuracy of impulse methods, SIAM J. Sci. Comput. 19 (4) (1998) 1290–1302.

[129] R. Cortez, A vortex/impulse method for immersed boundary motion in high reynolds number flows, J. Comput. Phys. 160 (2000) 385–400.

[130] A. Wiin-Nielsen, On the application of trajectory methods in numerical forecasting, Tellus 11 (1959) 180–196.

[131] J. S. Sawyer, A semi-Lagrangian method of solving the vorticity advection equation, Tellus 15 (1963) 336–342.

[132] A. J. Robert, A stable numerical integration scheme for the primitive meteorological equations, Atmos. Ocean 19 (1) (1981) 35–46.

[133] J. R. Bates, Semi-Lagrangian advective schemes and their use in meteorological modeling, Lectures in Applied Mathematics 22 (1985) 1–29.

[134] J. Stam, Stable fluids, in: Proceedings of the 26th annual conference on Computer graphics and interactive techniques, ACM Press/Addison-Wesley Publishing Co., 1999, pp. 121–128.

[135] A. V. Malevsky, Spline-characteristic method for simulation of convective turbulence, J. Comput. Phys. 123 (1996) 466–475.

[136] D. G. Dritschel, M. H. Ambaum, A contour-advective semi-Lagrangian numerical algorithm for simulating fine-scale conservative dynamical fields, Q. J. R. Meteorol. Soc. 123 (1997) 1097–1130.

[137] A. D. Kotelnikov, N. J. Zabusky, Vortex dynamics of a twice-accelerated interface in an incompressible ideal fluid, Astrophysical Journal Supplement Series 127 (2000) 389–394.

[138] M. Napolitano, L. A. Catalano, A multigrid solver for the vorticity-velocity Navier-Stokes equations, Intl. J. Num. Meth. Fluids 13 (1991) 49–59.

[139] G. Tryggvason, J. Abdollahi-Alibeik, W. W. Willmarth, A. Hirsa, Collision of a vortex pair with a contaminated free surface, Phys. Fluids A 4 (6) (1992) 1215–1229.

[140] W. E, J.-G. Liu, Finite difference methods for 3D viscous incompressible flows in the vorticity-vector potential formulation on nonstaggered grids, J. Comput. Phys. 138 (1997) 57–82.

[141] W. E, J.-G. Liu, Finite difference schemes for incompressible flows in the velocity-impulse density formulation, J. Comput. Phys. 130 (1997) 67–76.

[142] A. J. Chorin, Vortex sheet approximation of boundary layers, J. Comput. Phys. 27 (1978) 428–442.

[143] C. Airiau, A. Giovannini, Vorticity evolution on a separated wavy wall flow, ESAIM Proc. 7 (1999) 1–11.

[144] P. S. Bernard, Turbulent flow modeling using a fast, parallel, vortex tube and sheet method, ESAIM Proc. 7 (1999) 46–55.

[145] T. Schlick, Molecular Modeling and Simulation: An interdisciplinary guide, Springer, New York, NY, 2002.

[146] J. A. Sethian, J.-P. Brunet, A. Greenberg, J. P. Mesirov, Vortex methods and massively parallel processors, in: C. Anderson, C. Greengard (Eds.), Vortex Dynamics and Vortex Methods, no. 28 in Lectures in Applied Mathematics, American Mathematical Society, 1991, pp. 481–546.

[147] A. Gharakhani, A. F. Ghoniem, Massively parallel implementation of a 3D vortex-boundary element method, ESAIM Proc. 1 (1996) 213–223.

[148] A. R. Low, Postulates of hydrodynamics, Nature 121 (3050) (1928) 576.

[149] G. Tryggvason, Simulation of vortex sheet roll-up by vortex methods, J. Comput. Phys. 80 (1989) 1–16.

[150] D. W. Moore, The spontaneous appearance of a singularity in the shape of an evolving vortex sheet, Proc. Roy. Soc. London Ser. A 365 (1979) 105.

[151] R. Krasny, A study of singularity formation in a vortex sheet by the point-vortex approximation, J. Fluid Mech. 167 (1986) 65.

[152] T. Ishihara, Y. Kaneda, Singularity formation in three dimensional motion of a vortex sheet, J. Fluid Mech. 300 (1995) 339–366.

[153] T. Ishihara, Y. Kaneda, Singularity formation in the shape of a vortex sheet in three dimensions—numerical simulation, ESAIM Proc. 1 (1996) 463–479.

[154] M. Nitsche, Singularity formation in a cylindrical and a spherical vortex sheet, J. Comput. Phys. 173 (2001) 208–230.

[155] N. J. Zabusky, M. H. Hughes, K. V. Roberts, Contour dynamics for the Euler equations in two dimensions, J. Comput. Phys. 30 (1979) 96–106.

[156] T. F. Buttke, The observation of singularities in the boundary patches of constant vorticity, Phys. Fluids A 1 (7) (1989) 1283–1285.

[157] H. Kudela, Numerical study of the motion and deformation of two-dimensional bubbles by a vortex method, in: J. T. B. et al. (Ed.), Vortex Flows and Related Numerical Methods, Kluwer Academic Publishers, 1993, pp. 191–205.

[158] M. Nitsche, Self-similar shedding of vortex rings, J. Fluid Mech. 435 (2001) 397–407.

[159] G. Tryggvason, W. J. A. Dahm, K. Sbeih, Fine structure of vortex sheet roll-up by viscous and inviscid simulation, J. Fluids Eng. 113 (1991) 31–36.

[160] J. M. Nitsche, R. Krasny, A numerical study of vortex ring formation at the edge of a circular tube, J. Fluid Mech. 276 (1994) 139–161.

[161] P. Luchini, R. Tognaccini, Comparisons of a viscous and inviscid numerical simulations of the start-up vortex issuing from a semi-infinite flat plate, ESAIM Proc. 7 (1999) 247–257.

[162] C. K. Birdsall, D. Fuss, Clouds-in-clouds, clouds-in-cells physics for many-body plasma simulations, J. Comput. Phys. 3 (1969) 494–511.

[163] F. H. Harlow, The Particle-in-Cell method for two dimensional hydrodynamic problems, Tech. Rep. Report No. LAMS-2082, Los Alamos Scientific Laboratory (1956).

[164] W. E, J.-G. Liu, Finite difference schemes for incompressible flows in vorticity formulations, ESAIM Proc. 1 (1996) 181–195.

[165] A. K. Wong, J. A. Reizes, The vector potential in the numerical solution of three-dimensional fluid dynamics problems in multiply connected regions, J. Comput. Phys. 62 (1986) 124–142.

[166] J. P. Christiansen, Numerical simulation of hydrodynamics by the method of point vortices, J. Comput. Phys. 13 (1973) 363–379.

[167] J. P. Christiansen, N. J. Zabusky, Instability, coalescence and fission of finite-area vortex structures, J. Fluid Mech. 61 (1973) 219–243.

[168] J. P. Christiansen, VORTEX—a two-dimensional hydrodynamics simulation code, UKAEA Culham Lab. Rep. CLM-R106, UKAEA Culham (1970).

[169] G. J. Hirasaki, J. D. Hellums, Boundary conditions on the vector and scalar potentials in viscous three-dimensional hydrodynamics, Qu. App. Math. 28 (1970) 293–296.

[170] I. Zawadski, H. Aref, Mixing during vortex ring collision, Phys. Fluids A 3 (5) (1991) 1405–1410.

[171] R. Savoie, Y. Gagnon, Y. Mercadier, Numerical simulation of the starting flow down a step, ESAIM Proc. 1 (1996) 377–386.

[172] R. Abdolhosseini, R. Milane, Prediction of growth of turbulence in a uniformly sheared flow using the vortex-in-cell method, ESAIM Proc. 1 (1996) 387–400.

[173] C. R. Anderson, A method of local corrections for computing the velocity field due to a distribution of vortex blobs, J. Comput. Phys. 62 (1986) 111–123.

[174] R. W. Hockney, S. P. Goel, J. W. Eastwood, Quiet high-resolution computer models of a plasma, J. Comput. Phys. 14 (1974) 148–158.

[175] J. H. Walther, G. Morgenthal, An influence matrix particle-particle particle-mesh algorithm with exact particle-particle correction, J. Comput. Phys. 184 (2003) 670–678.

[176] J. H. Walther, G. Morgenthal, An immersed interface method for the vortex-in-cell algorithm, J. Turb. 3 (039).

[177] M. Cheng, Y. T. Chew, S. C. Luo, A hybrid vortex method for flows over a bluff body, Intl. J. Num. Meth. Fluids 24 (1997) 253–274.

[178] H. Kudela, P. Regucki, The vortex-in-cell method for the study of three-dimensional vortex structures, in: K. Bajer, H. Moffat (Eds.), Tubes, Sheets, and Singularities in Fluid Dynamics, Kluwer Academic Publishers, 2002, pp. 191–205.

[179] A. W. Appel, An efficient program for many-body simulation, SIAM J. Sci. Stat. Comput. 6 (1985) 85.

[180] J. E. Barnes, P. Hut, A hierarchical $\mathcal{O}$(N log N) force calculation algorithm, Nature 324 (1986) 446.

[181] L. Greengard, V. Rokhlin, A fast algorithm for particle simulations, J. Comput. Phys. 73 (1987) 325–348.

[182] W. Dehnen, A hierarchical $\mathcal{O}(N)$ force calculation algorithm, J. Comput. Phys. 179 (2002) 27–42.

[183] C. R. Anderson, An implementation of the fast multipole method without multipoles, SIAM J. Sci. Stat. Comput. 13 (1992) 923–947.

[184] G.-H. Cottet, P. Koumoutsakos, Vortex Methods: Theory and Practice, Cambridge Univ. Press, Cambridge, UK, 1999.

[185] O. H. Hald, Convergence of vortex methods for euler's equations, III, SIAM J. Numer. Anal. 24 (3) (1987) 538–582.

[186] J. K. Salmon, Parallel heirarchical N-body methods, Ph.D. thesis, California Institute of Technology (1991).

[187] V. M. Fernandez, N. J. Zabusky, P. Liu, S. Bhatt, A. Gerasoulis, Filament surgery and temporal grid adaptivity extensions to a parallel tree code for simulation and diagnosis in 3D vortex dynamics, ESAIM Proc. 1 (1996) 197–211.

[188] L. Qian, M. Vezza, A vorticity-based method for incompressible unsteady viscous flows, J. Comput. Phys. 172 (2001) 515–542.

[189] K. T. Lim, Mega-molecular dynamics on highly parallel computers: methods and applications, Ph.D. thesis, California Institute of Technology (1995).

[190] J. H. Strickland, L. A. Gritzo, R. S. Baty, G. F. Homicz, S. P. Burns, Fast multipole solvers for three-dimensional radiation and fluid flow problems, ESAIM Proc. 7 (1999) 408–417.

[191] Q. X. Wang, Variable order revised binary treecode, J. Comput. Phys. 200 (2004) 192–210.

[192] P. Ploumhans, G. S. Winckelmans, J. K. Salmon, Vortex particles and tree codes: I. Flows with arbitrary crossing between solid boundaries and particle redistribution lattice; II. Vortex ring encountering a plane at an angle, ESAIM Proc. 7 (1999) 335–348.

[193] D. G. Dritschel, N. J. Zabusky, On the nature of vortex interactions and models in unforced nearly-inviscid two-dimensional turbulence, Phys. Fluids 8 (5) (1996) 1252–1256.

[194] B. Legras, D. G. Dritschel, The elliptical model of two-dimensional vortex dynamics. I: The basic state, Phys. Fluids A 3 (5) (1991) 845–854.

[195] D. G. Dritschel, B. Legras, The elliptical model of two-dimensional vortex dynamics. II: Disturbance equations, Phys. Fluids A 3 (5) (1991) 855–869.

[196] J. Katz, A. Plotkin, Low speed aerodynamics, McGraw-Hill, New York, 1991.

[197] J.-Z. Wu, A theory of three-dimensional interfacial vorticity dynamics, Phys. Fluids 7 (10) (1995) 2375–2395.

[198] J. Malarkey, A. G. Davies, Use of Routh's correction in the cloud-in-cell discrete vortex method, J. Comput. Phys. 181 (2002) 753–759.

[199] G.-H. Cottet, P. Poncet, Particle methods for direct numerical simulations of three-dimensional wakes, J. Turb. 3 (038).

[200] M. H. Akbari, S. J. Price, Simulation of dynamic stall for a NACA 0012 airfoil using a vortex method, J. Fluids Struct. 17 (2003) 855–874.

[201] G. R. Baker, The "cloud in cell" technique applied to the roll up of vortex sheets, J. Comput. Phys. 31 (1) (1979) 76–95.

[202] S. H. Brecht, J. R. Ferrante, Vortex-in-cell simulations of buoyant bubbles in three dimensions, Phys. Fluids A 1 (7) (1989) 1166–1191.

[203] S. H. Brecht, J. R. Ferrante, Vortex-in-cell calculations in three dimensions, Computer Physics Communications 58 (1-2) (1990) 25–54.

[204] A. Gharakhani, A. F. Ghoniem, Simulation of three-dimensional internal flows by the random vortex and boundary element methods, ESAIM Proc. 1 (1996) 33–47.

[205] G. Comini, M. Manzan, G. Cortella, Open boundary conditions for the streamfunction-vorticity formulation of unsteady laminar convection, Numerical Heat Transfer, Part B 31 (1997) 217–234.

[206] A. Sohankar, C. Norberg, L. Davidson, Low Reynolds-number flow around a square cylinder at incidence: study of blockage, onset of vortex shedding and outlet boundary condition, Intl. J. Num. Meth. Fluids 26 (1998) 39–56.

[207] J. Reinaud, L. Joly, P. Chassaing, Numerical simulation of a variable-density mixing-layer, ESAIM Proc. 7 (1999) 359–368.

[208] C. S. Peskin, Flow patterns around heart valves: A numerical method, J. Comput. Phys. 10 (1972) 252–271.

[209] A. L. Fogelson, J. P. Keener, Immersed interface methods for Neumann and related problems in two and three dimensions, SIAM J. Sci. Comput. 22 (5) (2000) 1630–1654.

[210] R. J. Leveque, Z. Li, The immersed interface method for elliptic equations with discontinuous coefficients and singular sources, SIAM J. Numer. Anal. 31 (4) (1994) 1091–1044.

[211] S. Deng, K. Ito, Z. Li, Three-dimensional elliptic solvers for interface problems and applications, J. Comput. Phys. 184 (2003) 215–243.

[212] A. Gilmanov, F. Sotiropoulos, E. Balaras, A general reconstruction algorithm for simulating flows with complex 3d immersed boundaries on Cartesian grids, J. Comput. Phys. 191 (2003) 660–669.

[213] S. Marella, S. Krishnan, H. Liu, H. Udaykumar, Sharp interface Cartesian grid method I: An easily implemented technique for 3d moving boundary computations, J. Comput. Phys. 210 (2005) 1–31.

[214] T. Ye, W. Shyy, J. N. Chung, A fixed-grid, sharp-interface method for bubble dynamics and phase change, J. Comput. Phys. 174 (2001) 781–815.

[215] R. P. Fedkiw, Coupling an Eulerian fluid calculation to a Lagrangian solid calculation with the Ghost Fluid method, J. Comput. Phys. 175 (2002) 200–224.

[216] J. Pacheco, A. Pacheco-Vega, T. Rodić, R. Peck, Numerical simulations of heat transfer and fluid flow problems using an immersed-boundary finite-volume method on nonstaggered grids, Numerical Heat Transfer, Part B 48 (2005) 1–24.

[217] J. Hess, A. Smith, Calculation of potential flows about arbitrary bodies, Prog. Aeronaut. Sci. 8 (1967) 1–138.

[218] N. R. Clarke, O. R. Tutty, Construction and validation of a discrete vortex method for the two-dimensional incompressible Navier-Stokes equations, Comput. Fluids 23 (6) (1994) 751–783.

[219] J.-Z. Wu, J.-M. Wu, Interactions between a solid surface and a viscous compressible flow field, J. Fluid Mech. 254 (1993) 183–211.

[220] J. T. Beale, A grid-based boundary integral method for elliptic problems in three dimensions, SIAM J. Numer. Anal. 42 (2) (2004) 599–620.

[221] G. R. Baker, J. T. Beale, Vortex blob methods applied to interfacial motion, J. Comput. Phys. 196 (2004) 233–258.

[222] H.S.Udaykumar, R.Mittal, P.Rampunggoon, A.Khanna, A sharp interface cartesian grid method for simulating flows with complex moving boundaries, J. Comput. Phys. 174 (2001) 345–380.

[223] C. Albuquerque, G.-H. Cottet, A vortex method for flows with slight density variation, Numer. Methods Partial Differential Eq. 20 (2004) 199–229.

[224] J. Strickland, V. Porter, G. Homicz, A. Gossler, A vortex code for flow over rigid or flexible bluff bodies, J. Turb. 3 (042).

[225] J. T. Beale, A. Majda, High order accurate vortex methods with explicit velocity kernels, J. Comput. Phys. 58 (1985) 188–208.

[226] J. H. Walther, P. Koumoutsakos, Three-dimensional vortex methods for particle-laden flows with two-way coupling, J. Comput. Phys. 167 (2001) 39–71.

[227] E. Vorozhtsov, On the influence of shape and dimensions of particles on the properties of the modified particle-in-cell method, Comput. Fluids 9 (1981) 313–326.

[228] P. Koumoutsakos, A. Leonard, High resolution simulations of the flow around an impulsively started cylinder using vortex methods, J. Fluid Mech. 296 (1995) 1–38.

[229] J. H. Williamson, Four cheap improvements to the particle-mesh code, J. Comput. Phys. 41 (1981) 256–269.

[230] W. L. Kruer, J. M. Dawson, The dipole expansion method for plasma simulation, J. Comput. Phys. 13 (1973) 114–129.

[231] J. J. Monaghan, Extrapolating B-splines for interpolation, J. Comput. Phys. 60 (1985) 253–262.

[232] J. J. Monaghan, Particle method for hydrodynamics, Comput. Phys. Rep. 3 (1985) 71–124.

[233] C. S. Peskin, Numerical analysis of blood flow in the heart, J. Comput. Phys. 25 (1977) 220–252.

[234] S. O. Unverdi, G. Tryggvason, A front-tracking method for viscous, incompressible, multi-fluid flows, J. Comput. Phys. 100 (1992) 25–37.

[235] D. Wee, A. F. Ghoniem, Modified interpolation kernels for treating diffusion and remeshing in vortex methods, J. Comput. Phys. unk (2005) unk.

[236] C. Fureby, On subgrid scale modeling in large eddy simulations of compressible fluid flow, Phys. Fluids 8 (5) (1996) 1301–1311.

[237] A. B. Ebiana, R. W. Bartholomew, Design considerations for numerical filters used in vortex-in-cell algorithms, Comput. Fluids 25 (1) (1996) 61–75.

[238] J. R. Mansfield, O. M. Knio, C. Meneveau, A dynamic LES scheme for the vorticity transport equation: formulation and *a priori* tests, J. Comput. Phys. 145 (1998) 693–730.

[239] K. Takeda, O. R. Tutty, D. A. Nicole, Parallel discrete vortex methods on commodity supercomputers; an investigation into bluff body far wake behaviour, ESAIM Proc. 7 (1999) 418–428.

[240] P. G. Saffman, D. I. Meiron, Difficulties with three-dimensional weak solutions for inviscid incompressible flow, Phys. Fluids 29 (8) (1986) 2373–2375.

[241] G. Pedrizzetti, Insight into singular vortex flows, Fluid Dyn. Res. 10 (1992) 101–115.

[242] J. D. Eldredge, A. Leonard, T. Colonius, A general deterministic treatment of derivatives in particle methods, J. Comput. Phys. 180 (2002) 686–709.

[243] M. S. Ingber, A vorticity method for the solution of natural convection flows in enclosures, Intl. J. Num. Meth. Heat & Fluid FLow 13 (6) (2003) 655–671.

[244] M. J. Stock, A regularized inviscid vortex sheet method for three dimensional flows with density interfaces, Ph.D. thesis, University of Michigan (2006).

[245] F. Noca, D. Shiels, D. Jeon, Measuring instantaneous fluid dynamic forces on bodies using only velocity fields and their derivatives, J. Fluids Struct. 11 (1997) 345.

[246] F. Noca, D. Shiels, D. Jeon, A comparison of methods for evaluating time-dependent fluid dynamic forces on bodies using only velocity fields and their derivatives, J. Fluids Struct. 13 (1999) 551.

[247] C. R. Anderson, C. Greengard, On vortex methods, SIAM J. Numer. Anal. 22 (3) (1985) 413–440.

[248] S. Shankar, L. van Dommelen, A new diffusion procedure for vortex methods, J. Comput. Phys. 127 (1996) 88–109.

[249] M. Kiya, K. Toyoda, H. Ishii, M. Kitamura, T. Ohe, Numerical simulation and flow visualization experiment on deformation of pseudo-elliptic vortex rings, Fluid Dyn. Res. 10 (1992) 117–131.

[250] A. J. Chorin, Numerical study of slightly viscous flow, J. Fluid Mech. 57 (1973) 785–796.

[251] O. H. Hald, Convergence of a random method with creation of vorticity, SIAM J. Sci. Stat. Comput. 7 (1986) 1373–1386.

[252] E. Puckett, A study of the vortex sheet method and its rate of convergence, SIAM J. Sci. Comput. 10 (2) (1989) 298–327.

[253] I. Mortazavi, P. Micheau, A. Giovannini, Numerical convergence of the random vortex method for complex flows, ESAIM Proc. 1 (1996) 521–538.

[254] Y. Gagnon, A. Giovannini, P. Hébrard, Numerical simulation and physical analysis of high Reynolds number recirculating flows behind sudden expansions, Phys. Fluids A 5 (10) (1993) 2377–2389.

[255] R. Abdolhosseini, R. E. Milane, Vortex-in-cell simulation of a spatially growing mixing layer, Mechanics Research Communications 25 (3) (1998) 237–244.

[256] J. S. Marshall, J. R. Grant, A Lagrangian vorticity collocation method for viscous, axisymmetric flows with and without swirl, J. Comput. Phys. 138 (1997) 302–330.

[257] R. E. Milane, Large eddy simulation (2d) using diffusion-velocity method and vortex-in-cell, Intl. J. Num. Meth. Fluids 44 (2004) 837–860.

[258] L. Greengard, The core spreading vortex method approximates the wrong equation, J. Comput. Phys. 61 (1985) 345–348.

[259] L. F. Rossi, Resurrecting core spreading vortex methods: A new scheme that is both deterministic and convergent, SIAM J. Sci. Comput. 17 (3) (1996) 370–397.

[260] G. Tryggvason, W. J. A. Dahm, An integral method for mixing, chemical reactions, and extinction in unsteady strained diffusion layers, Combustion and Flame 83 (1991) 207–220.

[261] C. H. H. Chang, W. J. A. Dahm, G. Tryggvason, Lagrangian model simulations of molecular mixing, including finite rate chemical reactions, in a temporally developing shear layer, Phys. Fluids A 3 (5) (1991) 1300–1311.

[262] D. Beigie, A. Leonard, S. Wiggins, A global study of enhanced stretching and diffusion in chaotic tangles, Phys. Fluids A 3 (5) (1991) 1039–1050.

[263] R. P. Feynman, Application of quantum dynamics to liquid helium, Prog. Low Temp Phys. I (1957) 16–53.

[264] A. J. Chorin, Hairpin removal in vortex interactions, J. Comput. Phys. 91 (1990) 1–21.

[265] A. J. Chorin, Hairpin removal in vortex interactions II, J. Comput. Phys. 107 (1993) 1–9.

[266] A. J. Chorin, Microstructure, renormalization, and more efficient vortex methods, ESAIM Proc. 1 (1996) 1–14.

[267] S. Mas-Gallic, Contribution à l'analyse numérique des méthodes particulaires, Ph.D. thesis, Université Paris VI (1987).

[268] P. Degond, S. Mas-Gallic, The weighted particle method for convection-diffusion equations, Part 1: The case of an isotropic viscosity, Math. Comput. 53 (188) (1989) 485–507.

[269] P. Degond, S. Mas-Gallic, The weighted particle method for convection-diffusion equations, Part 2: The anisotropic case, Math. Comput. 53 (188) (1989) 509–525.

[270] D. Fishelov, A new vortex scheme for viscous flows, J. Comput. Phys. 86 (1990) 211–224.

[271] J. McNames, A fast nearest neighbor algorithm based on a principal axis search tree, IEEE Transactions on Pattern Analysis and Machine Intelligence 23 (9) (2001) 964–976.

[272] I. Lakkis, A. F. Ghoniem, Grid-free simulation of radiative transport in a participating medium, ESAIM Proc. 7 (1999) 234–246.

[273] A. Gharakhani, A higher order vorticity redistribution method for 3-D diffusion in free space, Tech. Rep. Report No. SAND2000-2505, Sandia National Laboratories (2000).

[274] A. Gharakhani, Application of VRM to LES of incompressible flow, J. Turb. 4 (004).

[275] M. J. Fritts, W. P. Crowley, H. E. Trease (Eds.), The Free Lagrange Method, no. 238 in Lecture Notes in Physics, Springer-Verlag, 1985.

[276] M. Börgers, C. S. Peskin, A lagrangian fractional step method for the incompressible Navier-Stokes equations on a periodic domain, J. Comput. Phys. 70 (1987) 397.

[277] G. Russo, A Lagrangian vortex method for the incompressible Navier-Stokes equations, in: C. Anderson, C. Greengard (Eds.), Vortex Dynamics and Vortex Methods, no. 28 in Lectures in Applied Mathematics, American Mathematical Society, 1991, pp. 585–596.

[278] J. M. R. Graham, Computation of viscous separated flow using a particle method, in: Numerical methods for fluid dynamics III; Proceedings of the Conference, Oxford, England, Mar. 21-24, 1988 (A89-51531 22-34). Oxford/New York, Clarendon Press/Oxford University Press, 1988, p. 310-317., 1988, pp. 310–317.

[279] Z. Y. Lu, T. Ross, Diffusing-vortex numerical scheme for solving incompressible Navier-Stokes equation, J. Comput. Phys. 95 (1991) 400–435.

[280] S. Ghosal, P. Moin, The basic equations for the large eddy simulation of turbulent flows in complex geometry, J. Comput. Phys. 118 (1995) 24–37.

[281] G.-H. Cottet, Artificial viscosity methods for vortex and particle methods, J. Comput. Phys. 127 (1996) 299–308.

[282] G.-H. Cottet, D. Jiroveanu, B. Michaux, Vorticity dynamics and turbulence models for large-eddy simulation, Math. Model. Numer. Anal. 37 (1) (2003) 187–207.

[283] T. S. Lundgren, J. Yao, N. N. Mansour, Microburst modelling and scaling, J. Fluid Mech. 239 (1992) 461–488.

[284] S. M. Scorpio, Fully nonlinear ship-wave computations using a multipole accelerated, desingularized method, Ph.D. thesis, University of Michigan, Ann Arbor, MI (1997).

[285] M. J. Lighthill, in: L. Rosenhead (Ed.), Laminar boundary layers, Oxford Univ. Press, Cambridge, UK, 1963, pp. 46–113.

[286] A. Leonard, I dunno, in: I dunno, no. 59 in Lecture Notes in Physics, Springer-Verlag, 1976, pp. 280–284.

[287] P. Koumoutsakos, A. Leonard, F. Pépin, Boundary conditions for viscous vortex methods, J. Comput. Phys. 113 (1994) 52.

[288] J. Szumbarski, P. Wald, Stochastic vortex simulation of an unsteady viscous flow in a multiconnected domain, ESAIM Proc. 1 (1996) 153–167.

[289] F. Buron, L. Pérault, A new approach to the particle-in-cell method for the computation of three-dimensional incompressible unsteady flows, ESAIM Proc. 7 (1999) 80–93.

[290] N. J. Zabusky, Vortex paradigm for accelerated inhomogenous flows: Visiometrics for the Rayleigh-Taylor and Richtmyer-Meshkov environments, Ann. Rev. Fluid Mech. 31 (1999) 495–536.

[291] G. Tryggvason, H. Aref, Numerical experiments on Hele-Shaw flow with a sharp interface, J. Fluid Mech. 136 (1983) 1–30.

[292] H. Aref, G. Tryggvason, Vortex dynamics of passive and active interfaces, Physica D 12 (1984) 59–70.

[293] G. de Josselin de Jong, Singularity distributions for the analysis of multiple-fluid flow through porous media, J. Geophysics Res. 65 (11) (1960) 3739–3758.

[294] G. Tryggvason, H. Aref, Finger-interaction mechanisms in stratified Hele-Shaw flow, J. Fluid Mech. 154 (1985) 287–301.

[295] T. Y. Hou, J. S. Lowengrub, M. J. Shelley, Removing the stiffness from interfacial flows with surface tension, J. Comput. Phys. 114 (1994) 312–338.

[296] J. A. Zufiria, Vortex-in-cell simulation of bubble competition in a Rayleigh-Taylor instability, Phys. Fluids 31 (11) (1988) 3199–3212.

[297] G. R. Baker, D. I. Meiron, S. A. Orszag, Generalized vortex methods for free-surface flow problems, J. Fluid Mech. 123 (1982) 477–501.

[298] J. C. S. Meng, The physics of vortex-ring evolution in a stratified and shearing environment, J. Fluid Mech. 84 (3) (1978) 455–469.

[299] C. R. Anderson, A vortex method for flows with slight density variations, J. Comput. Phys. 61 (1985) 417–444.

[300] W. J. A. Dahm, C. M. Scheil, G. Tryggvason, Dynamics of vortex interaction with a density interface, J. Fluid Mech. 205 (1989) 1–43.

[301] M. S. Longuet-Higgins, E. D. Cokelet, The deformation of steep surface waves on water I. A numerical method of computation, Proc. Roy. Soc. London A 350 (1976) 1–26.

[302] D. I. Pullin, Numerical studies of surface-tension effects in non-linear Kelvin-Helmholtz and Rayleigh-Taylor instability, J. Fluid Mech. 119 (1982) 507–532.

[303] R. M. Kerr, Simulation of Rayleigh-Taylor flows using vortex blobs, J. Comput. Phys. 76 (1988) 48–84.

[304] M. Nitsche, P. H. Steen, Numerical simulations of inviscid capillary pinchoff, J. Comput. Phys. 200 (2004) 299–324.

[305] S. J. Zaroodny, M. D. Greenberg, On a vortex sheet approach to the numerical calculation of water waves, J. Comput. Phys. 11 (1973) 440–446.

[306] M. S. Longuet-Higgins, E. D. Cokelet, The deformation of steep surface waves on water II. Growth of normal-mode instabilities, Proc. Roy. Soc. London A 364 (1978) 1–28.

[307] G. Tryggvason, S. O. Unverdi, M. Song, J. Abdollahi-Alibeik, Interaction of vortices with a free surface and density interfaces, in: C. Anderson, C. Greengard (Eds.), Vortex Dynamics and Vortex Methods, no. 28 in Lectures in Applied Mathematics, American Mathematical Society, 1991, pp. 679–699.

[308] L. Chen, W. S. Vorus, Application of a vortex method to free surface flows, Intl. J. Num. Meth. Fluids 14 (1992) 1289–1310.

[309] E. P. Rood, Interpreting vortex interactions with a free surface, J. Fluids Eng. 116 (1994) 91–94.

[310] D. G. Dommermuth, The initialization of vortical free-surface flows, J. Fluids Eng. 116 (1994) 95–102.

[311] X. Zhang, A. F. Ghoniem, Transport element method for axisymmetric variable-density flow and its application to the spread and dispersion of a dense cloud, ESAIM Proc. 1 (1996) 77–94.

[312] T. Y. Hou, J. S. Lowengrub, M. J. Shelley, Boundary integral methods for multicomponent fluids and multiphase materials, J. Comput. Phys. 169 (2001) 302–362.

[313] Y.-N. Young, F. E. Ham, M. Herrmann, N. N. Mansour, Interaction between turbulent flow and free surfaces, in: Annual Research Briefs, Center for Turbulence Research, 2002, pp. 301–311.

[314] C. Song, A. I. Serviente, A numerical study of breaking waves, Phys. Fluids 16 (7) (2004) 2649–2667.

[315] D. Lörstad, M. Francois, W. Shyy, L. Fuchs, Assessment of volume of fluid and immersed boundary methods for droplet computations, Intl. J. Num. Meth. Fluids 46 (2004) 109–125.

[316] A. Esmaeeli, G. Tryggvason, A front tracking method for computations of boiling in complex geometries, Intl. J. Multiphase Flow 30 (2004) 1037–1050.

[317] N. Al-Rawahi, G. Tryggvason, Numerical simulation of dendritic solidification with convection: Three-dimensional flow, J. Comput. Phys. 194 (2004) 677–696.

[318] F. S. de Sousa, M. Mangiavacchi, L. G. Nonato, A. Castelo, M. F. Tomé, V. G. Ferreira, J. A. Cuminato, S. McKee, A front-tracking/front-capturing method for the simulation of 3D multi-fluid flows with free surfaces, J. Comput. Phys. 198 (2004) 469–499.

[319] C. Y. Shen, T. E. Evans, A free-surface hydrodynamic model for density-stratified flow in the weakly to strongly non-hydrostatic regime, J. Comput. Phys. 200 (2004) 695–717.

[320] T. Y. Hou, J. S. Lowengrub, M. J. Shelley, The long-time motion of vortex sheets with surface tension, Phys. Fluids 9 (7) (1997) 1933–1954.

[321] B. Perot, R. Nallapati, A moving unstructured staggered mesh method for the simulation of incompressible free-surface flows, J. Comput. Phys. 184 (2003) 192–214.

[322] S. Popinet, S. Zaleski, A front-tracking algorithm for accurate representation of surface tension, Intl. J. Num. Meth. Fluids 30 (1999) 775–793.

[323] J. N. Chung, T. R. Troutt, Numerical models for two-phase turbulent flows, J. Fluid Mech. 186 (1996) 199–222.

[324] C. T. Crowe, T. R. Troutt, J. N. Chung, Numerical models for two-phase turbulent flows, Ann. Rev. Fluid Mech. 28 (1996) 11–43.

[325] A. Esmaeeli, G. Tryggvason, Direct numerical simulations of bubbly flows Part 2. Moderate Reynolds number arrays, J. Fluid Mech. 385 (1999) 325–358.

[326] H. H. Hu, N. A. Patankar, M. Y. Zhu, Direct numerical simulations of fluid-solid systems using the arbitrary Lagrangian-Eulerian technique, J. Comput. Phys. 169 (2001) 427–462.

[327] E. Wallner, E. Meiburg, Numerical simulations of two-way coupled particle laden mixing layers, ESAIM Proc. 7 (1999) 429–439.

[328] K. D. Squires, J. K. Eaton, Preferential concentration of particles by turbulence, Phys. Fluids A 3 (5) (1991) 1169–1178.

[329] S. B. Pope, Turbulent Flows, Cambridge Univ. Press, Cambridge, UK, 2000.

[330] M. Kiya, M. Arie, Helmholts instability of a vortex sheet in uniform shear flow, Phys. Fluids 22 (1979) 378–379.

[331] J.-M. Baey, P. Rivière, X. Carton, Ocean jet instability: a model comparison, ESAIM Proc. 7 (1999) 12–23.

[332] N. Peters, P. Trouillet, On the role of quasi-one-dimensional dissipation layers in turbulent scalar mixing, in: Annual Research Briefs, Center for Turbulence Research, 2002, pp. 27–40.

[333] J. Glimm, O. McBryan, R. Menikoff, D. H. Sharp, Front-tracking applied to Rayleigh-Taylor instability, SIAM J. Sci. Stat. Comput. 7 (1) (1986) 230–251.

[334] A. F. Ghoniem, O. M. Knio, The development and application of the transport element method to three dimensional reacting shear layers, in: C. Anderson, C. Greengard (Eds.), Vortex Dynamics and Vortex Methods, no. 28 in Lectures in Applied Mathematics, American Mathematical Society, 1991, pp. 165–218.

[335] W. J. A. Dahm, G. Tryggvason, R. Frederiksen, M. Stock, Local Integral Moment (LIM) simulations, in: C. Baukal (Ed.), Computational Fluid Dynamics in Industrial Combustion, CRC Press, 2000, Ch. 5, pp. 161–208.

[336] J. Han, G. Tryggvason, W. J. A. Dahm, LIM-based modelling of multiphase flow problems for application to Chevron's wet gas production systems, final report (2000).

[337] A. F. Ghoniem, O. M. Knio, A. Krishnan, Lagrangian simulation of the early stages of a reacting jet, in: Twenty-Third Symposium (International) on Combustion, The Combustion Institute, 1990, pp. 699–705.

[338] M. Soteriou, A. F. Ghoniem, Numerical simulation of unsteady combustion using the transport element method, ESAIM Proc. 1 (1996) 429–446.

[339] I. Lakkis, A. F. Ghoniem, Axisymmetric vortex method for low-mach number, diffusion-controlled combustion, J. Comput. Phys. 184 (2003) 435–475.

[340] J. Jiménez, C. Martel, Fractal interfaces and product generation in the two-dimensional mixing layer, Phys. Fluids A 3 (5) (1991) 1261–1268.

[341] S. Mas-Gallic, M. Louaked, O. Pironneau, A particle in cell method for the 2-D compressible Euler equations, in: J. T. B. et al. (Ed.), Vortex Flows and Related Numerical Methods, Kluwer Academic Publishers, 1993, pp. 373–387.

[342] A. Gharakhani, A. F. Ghoniem, 3D vortex simulation of flow in an opposed-piston engine, ESAIM Proc. 7 (1999) 161–172.

[343] J. D. Eldredge, T. Colonius, A. Leonard, A dilating vortex particle method for compressible flow, J. Turb. 3 (036).

[344] J. D. Eldredge, T. Colonius, A. Leonard, A vortex particle method for two-dimensional compressible flow, J. Comput. Phys. 179 (2002) 371–399.

[345] M. Nitsche, J. H. Strickland, Extension of the gridless vortex method into the compressible flow regime, J. Turb. 3 (050).

[346] F. Thirifay, G. Winckelmans, Development of a lagrangian method for combustion and application to the planar methand-air jet diffusion flame, J. Turb. 3 (059).

[347] P. Koumoutsakos, Inviscid axisymmetrization of an elliptical vortex, J. Comput. Phys. 138 (1997) 821–857.

[348] G. Birkhoff, Helmholtz and Taylor instability, in: Proceedings, Symp. Appl. Math., 13th, Amer. Math. Soc., 1962, pp. 55–76.

[349] A. I. van de Vooren, A numerical investigation of the rolling-up of vortex sheets, Proc. Roy. Soc. London A 373 (1752) (1980) 67–91.

[350] A. Michalke, On spatially growing disturbances in an inviscid shear layer, J. Fluid Mech. 23 (3) (1965) 521–544.

[351] J. C. Lasheras, H. Choi, Three-dimensional instability of a plane free shear layer: an experimental study of the formation and evolution of streamwise vortices, J. Fluid Mech. 189 (1988) 53–86.

[352] T. Dallard, F. K. Browand, The growth of large scales at defect sites in the plane mixing layer, J. Fluid Mech. 247 (1993) 339–368.

[353] R. W. Metcalfe, S. A. Orszag, M. E. Brachet, S. Menon, J. J. Riley, Secondary instability of a temporally growing mixing layer, J. Fluid Mech. 184 (1987) 207–243.

[354] R. D. Moser, M. M. Rogers, The three-dimensional evolution of a plane mixing layer: pairing and transition to turbulence, J. Fluid Mech. 247 (1993) 275–320.

[355] M. M. Rogers, R. D. Moser, Spanwise scale selection in plane mixing layers, J. Fluid Mech. 247 (1993) 321–337.

[356] W. T. Ashurst, Numerical simulation of turbulent mixing layers via vortex dynamics, Tech. rep., Sandia Laboratories, Livermore, CA, SAND77-8613 (Feb. 1977).

[357] J.-G. Liu, C.-W. Shu, A high-order discontinuous Galerkin method for 2d incompressible flows, J. Comput. Phys. 160 (2000) 577–596.

[358] R. T. Pierrehumbert, S. E. Widnall, The two- and three-dimensional instabilities of a spatially periodic shear layer, J. Fluid Mech. 114 (1982) 59–82.

[359] P. T. Fink, W. K. Soh, A new approach to roll-up calculations of vortex sheets, Proc. Roy. Soc. London A 362 (1709) (1978) 195–209.

[360] R. Krasny, M. Nitsche, The onset of chaos in vortex sheet flow, J. Fluid Mech. 454 (2002) 47–69.

[361] S. Kida, M. Takaoka, F. Hussain, Collision of two vortex rings, J. Fluid Mech. 230 (1991) 583–646.

[362] J. S. Turner, Buoyant vortex rings, Proc. Roy. Soc. London A 239 (1216) (1957) 61–75.

[363] L. E. Fraenkel, Examples of steady vortex rings of small cross-section in an ideal fluid, J. Fluid Mech. 51 (1972) 119–135.

[364] J. Norbury, A family of steady vortex rings, J. Fluid Mech. 57 (1973) 417–431.

[365] B. J. Cantwell, N. Rott, The decay of a viscous vortex pair, Phys. Fluids 31 (11) (1988) 3213–3224.

[366] K. B. Southerland, J. R. P. III, W. J. A. Dahm, K. A. Buch, An experimental study of the molecular mixing process in an axisymmetric laminar vortex ring, Phys. Fluids A 3 (5) (1991) 1385–1392.

[367] J. O. Dabiri, M. Gharib, Fluid entrainment by isolated vortex rings, J. Fluid Mech. 511 (2004) 311–331.

[368] K. Shariff, Vortex rings, Ann. Rev. Fluid Mech. 24 (1992) 235–279.

[369] T. Maxworthy, Turbulent vortex rings, J. Fluid Mech. 64 (2) (1974) 227–239.

[370] M. Nitsche, Evolution of a cylindrical and a spherical vortex sheet, ESAIM Proc. 1 (1996) 401–414.

[371] Q. Nie, G. Baker, Application of adaptive quadrature to axi-symmetric vortex sheet motion, J. Comput. Phys. 143 (1998) 49–69.

[372] M. Nitsche, Numerical simulation of axisymmetric vortex sheet roll-up, in: J. T. B. et al. (Ed.), Vortex Flows and Related Numerical Methods, Kluwer Academic Publishers, 1993, pp. 293–301.

[373] M. Kiya, Y. Ido, H. Akiyama, Vortical structure in forced unsteady circular jet: Simulation by 3D vortex method, ESAIM Proc. 1 (1996) 503–520.

[374] W. T. Ashurst, Large eddy simulation via vortex dynamics, in: Presented at the AIAA Comput. Fluid Dyn. Conf., 6th, 1983, AIAA Pap. No. 83-1879-CP.

[375] D. E. Parekh, A. Leonard, W. C. Reynolds, Insight into singular vortex flows, Bull. Am. Phys. Soc. 28 (1983) 1353.

[376] D. Auerbach, Stirring properties of vortex rings, Phys. Fluids A 3 (5) (1991) 1351–1355.

[377] W. C. Reynolds, D. E. Parekh, P. J. D. Juvet, M. J. D. Lee, Bifurcating and blooming jets, Ann. Rev. Fluid Mech. 35 (2003) 295–315.

[378] W. J. A. Dahm, C. E. Frieler, G. Tryggvason, Vortex structure and dynamics in the near field of a coaxial jet, J. Fluid Mech. 241 (1992) 371–402.

[379] L. L. Yuan, R. L. Street, Trajectory and entrainment of a round jet in crossflow, Phys. Fluids 10 (9) (1998) 2323–2335.

[380] F. F. Grinstein, C. R. DeVore, Dynamics of coherent structures and transition to turbulence in free square jets, Phys. Fluids 8 (5) (1996) 1237–1251.

[381] A. Leonard, Vortex simulation of three-dimensional spotlike disturbances in a laminar boundary layer, in: L. J. S. B. et al. (Ed.), Turbulent Shear Flows II, Springer-Verlag, 1980, pp. 67–77.

[382] R. Krasny, Computation of vortex sheet roll-up in the trefftz plane, J. Fluid Mech. 184 (1987) 123–155.

[383] M. Mokry, A. Yeung, On the evolution of trailing vortex sheets, ESAIM Proc. 7 (1999) 292–303.

[384] J. S. Marshall, Response of a columnar vortex to a wrapped vortex loop, ESAIM Proc. 7 (1999) 280–391.

[385] B. Couët, Evolution of turbulence by three-dimensional numerical particle-vortex tracing, Ph.D. thesis, Stanford University Institute for Plasma Research (1979).

[386] M. J. Graham, Q. Zhang, Numerical simulations of deep nonlinear Richtmyer-Meshkov instability, Astrophysical Journal Supplement Series 127 (2000) 339–346.

[387] N. J. Zabusky, A. D. Kotelnikov, Y. Gulak, G. Peng, Amplitude growth rate of a Richtmyer-Meshkov unstable two-dimensional interface to intermediate times, J. Fluid Mech. 475 (2003) 147–162.

[388] J.-F. Haas, B. Sturtevant, Interaction of weak shock waves with cylindrical and spherical gas inhomogeneities, J. Fluid Mech. 181 (1987) 41–76.

[389] P. F. Linden, The interaction of a vortex ring with a sharp density interface: a model for turbulent entrainment, J. Fluid Mech. 60 (1973) 467–480.

[390] S. S. Shy, R. E. Breidenthal, Turbulent stratified interfaces, Phys. Fluids A 3 (5) (1991) 1278–1285.

[391] G. Tryggvason, Deformation of a free surface as a result of vortical flows, Phys. Fluids 31 (1988) 955–957.

[392] W. W. Willmarth, G. Tryggvason, A. Hirsa, D. Yu, Vortex pair generation and interaction with a free surface, Phys. Fluids A 1 (1) (1989) 170–172.

[393] G. Tryggvason, Vortex dynamics of stratified flows, in: R. E. Caflisch (Ed.), Proceedings of the Workshop on Mathematical Aspect of Vortex Dynamics, Leesburg, Virginia, April 25-27, 1988, 1989, pp. 160–170.

[394] G. L. Brown, A. Roshko, On density effects and large structure in turbulent mixing layers, J. Fluid Mech. 64 (4) (1974) 775–816.

[395] N. Freed, D. Ofer, D. Shvarts, S. A. Orszag, Two-phase flow analysis of self-similar turbulent mixing by Rayleigh-Taylor instability, Phys. Fluids A 3 (5) (1991) 912–918.

[396] G. Tryggvason, S. O. Unverdi, Computations of the three-dimensional Rayleigh-Taylor instability, Phys. Fluids 2 (5) (1990) 656–659.

[397] D. L. Youngs, Three-dimensional numerical simulations of turbulent mixing by Rayleigh-Taylor instability, Phys. Fluids A 3 (5) (1991) 1312–1320.

[398] F. M. Hill, A numerical study of the descent of a vortex pair in a stably stratified atmosphere, J. Fluid Mech. 71 (1) (1975) 1–13.

[399] L. A. Dávalos-Orozco, J. E. Aguilar-Rosas, Rayleigh-Taylor instability of a continuously stratified fluid under a general rotation field, Phys. Fluids A 1 (7) (1989) 1192–1199.

[400] J. S. Turner, Buoyant plumes and thermals, Ann. Rev. Fluid Mech. 1 (1969) 29–44.

[401] M. Shusser, M. Gharib, A model for vortex ring formation in a starting buoyant plume, J. Fluid Mech. 416 (2000) 173–185.

[402] R. S. Scorer, Experiments on convection of isolated masses of buoyant fluid, J. Fluid Mech. 2 (1957) 583–594.

[403] J. D. Bugg, R. D. Rowe, Modeling the initial motion of large cylindrical and spherical bubbles, Intl. J. Num. Meth. Fluids 13 (1991) 109–129.

[404] T. B. Benjamin, Gravity currents and related phenomena, J. Fluid Mech. 31 (2) (1968) 209–248.

[405] T. S. Lundgren, N. N. Mansour, Vortex ring bubbles, J. Fluid Mech. 224 (1991) 177–196.

[406] C. P. Verdon, R. L. McCrory, R. L. Morse, G. R. Baker, D. I. Meiron, S. A. Orszag, Nonlinear effects of multifrequency hydrodynamic instabilities on ablatively accelerated thin shells, Phys. Fluids 25 (1982) 1653–1674.

[407] G. K. Batchelor, J. M. Nitsche, Instability of stationary unbounded stratified fluid, J. Fluid Mech. 227 (1991) 357–391.

[408] M. R. E. Proctor, A note on the nonlinear development of the Batchelor-Nitsche instability, J. Fluid Mech. 254 (1993) 313–321.

[409] C. Pozrikidis, Numerical simulation of three-dimensional bubble oscillations by a generalized vortex method, Theoret. Comput. Fluid Dynamics 16 (2001) 151–169.

[410] T. A. Johnson, V. C. Patel, Flow past a sphere up to a Reynolds number of 300, J. Fluid Mech. 378 (1999) 19–70.

[411] P. G. Saffman, Approach of a vortex pair to a rigid free surface in viscous fluid, Phys. Fluids A 3 (5) (1991) 984–985.

[412] C. H. Liu, A three-dimensional vortex particle-in-cell method for vortex motions in the vicinity of a wall, Intl. J. Num. Meth. Fluids 37 (5) (2001) 501–523.

[413] F. W. Roos, W. W. Willmarth, Some experimental results on sphere and disk drag, AIAA Journal 9 (1971) 285–291.

[414] G. Oddie, H. Shi, L. J. Durlofsky, K. Aziz, B. Pfeffer, J. A. Holmes, Experimental study of two and three phase flows in large diameter inclined pipes, Intl. J. Multiphase Flow 29 (2003) 527–558.

[415] R. I. Issa, M. H. W. Kempf, Simulation of slug flow in horizontal and nearly horizontal pipes with the two-fluid model, Intl. J. Multiphase Flow 29 (2003) 69–95.